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Mathematics. — Contribution à la théorie additive des nombres. Par
J. G. VAN DER CORPUT. (Troisième communication).

(Communicated at the meeting of April 30, 1938).

Lemme 6: Si nous désignons par α un nombre positif ≤ 1 et par t un entier ≥ 4 , nous avons

$$\left| \sum_{h=2}^{t-2} \frac{h^{-1+\alpha}}{\log(t-h)} - \frac{t^\alpha}{\alpha \log t} \right| < \frac{c_{49} t^\alpha}{(\log t)^2},$$

où c_{49} ne dépend que de α .

Démonstration: Nous avons

$$\begin{aligned} \sum_{2 \leq h \leq t-\sqrt{t}} h^{-1+\alpha} \frac{\log t - \log(t-h)}{\log(t-h)} & \\ & \equiv \sum_{2 \leq h \leq t-\sqrt{t}} \left(\frac{h+1}{2} \right)^{-1+\alpha} \frac{\log t - \log(t-h)}{\frac{1}{2} \log t} \\ & < \frac{2^{2-\alpha}}{\log t} \int_0^t u^{-1+\alpha} \left(\log \frac{t}{t-u} \right) du \\ & = \frac{2^{2-\alpha} t^\alpha}{\log t} \int_0^1 v^{-1+\alpha} \left(\log \frac{1}{1-v} \right) dv, \\ \sum_{t-\sqrt{t} < h \leq t-2} h^{-1+\alpha} \frac{\log t - \log(t-h)}{\log(t-h)} & \leq \sum_{t-\sqrt{t} < h \leq t-2} \left(\frac{1}{2} t \right)^{-1+\alpha} \frac{\log t}{\log 2} \\ & < 2^{1-\alpha} t^{-\frac{1}{2}+\alpha} \frac{\log t}{\log 2} \end{aligned}$$

et

$$\left| \sum_{h=2}^{t-2} h^{-1+\alpha} - \frac{t^\alpha}{\alpha} \right| < c_{50},$$

c_{50} désignant un nombre dépendant uniquement de α . Le lemme est donc une conséquence de l'identité

$$\sum_{h=2}^{t-2} \frac{h^{-1+\alpha}}{\log(t-h)} = \sum_{h=2}^{t-2} \frac{h^{-1+\alpha}}{\log t} + \frac{1}{\log t} \sum_{h=2}^{t-2} h^{-1+\alpha} \frac{\log t - \log(t-h)}{\log(t-h)}.$$

Comme je l'ai déjà fait remarquer dans la deuxième communication, il est possible de déduire les propositions 3 et 4 de la proposition 2,

mais après avoir écrit la communication précédente, j'ai constaté que dans la proposition 2 la condition, exprimée par l'inégalité (22), est superflue, c'est-à-dire, au lieu de la proposition 2, j'ai également à ma disposition le théorème suivant.

Proposition 5: *Supposons que les conditions de la proposition 1 soient vérifiées, que la fonction $H(q, t)$ possède la propriété multiplicative, exprimée par (18), et qu'elle satisfasse pour tout nombre naturel q , tout entier t et tout nombre naturel m à l'inégalité*

$$|H(q, t)| \leq \gamma_m q^{-1 + \frac{1}{m}} \quad (32)$$

Sous ces conditions on a pour tout nombre positif m et pour tout nombre $\omega > m$

$$\sum_{t=2}^{[N]} |L(t) - \Lambda(t) \prod_p \sum_{\xi=0}^{\left[\frac{\omega \log \log t}{\log p}\right]} H(p^\xi, t)|^2 < c_{51} I^2 I'^2 N^3 n^{-m},$$

où c_{51} dépend uniquement de m, l, ω et des suites (γ) et (η) .

Démonstration: La démonstration est analogue à celle de la proposition 2. Nous appliquons la proposition 1, où m est remplacé par le plus petit entier $\equiv m + 1$, de sorte que nous trouvons un entier $\sigma \equiv m + 1$, dépendant uniquement de m et de la suite (η) , et aussi un nombre c_{19} , dépendant uniquement de m, l et des suites (γ) et (η) , tels que la seconde formule, figurant à la page 355, soit vérifiée. De la même manière que dans la démonstration de la proposition 2 pour le résultat analogue nous observons qu'il suffit de déduire les inégalités

$$\sum_{\sqrt{N} < t \leq N} \left| \sum_q \tau H(q, t) \right|^2 < c_{52} N n^{-m} \quad (\tau = 6, 7 \text{ et } 8); \quad (33)$$

$\sum_q$ est étendu à tous les nombres naturels $q \leq n^\tau$ possédant au moins un diviseur $p^\xi > (\log t)^\omega$; $\sum_q$ est étendu à tous les entiers $q > n^\tau$ et

$\leq N^{\frac{1}{4l+7}}$ dont tous les diviseurs de la forme p^ξ sont $\leq (\log t)^\omega$, tandis

que $\sum_q$ est étendu à tous les entiers $q > N^{\frac{1}{4l+7}}$ dont tous les diviseurs de la forme p^ξ sont $\leq (\log t)^\omega$; dans cette démonstration $c_{52}, c_{53}, \dots, c_{65}$ dépendent uniquement de m, l, ω et des suites (γ) et (η) .

On a

$$\sum_{\sqrt{N} < t \leq N} \left| \sum_q \tau H(q, t) \right|^2 = \sum_{\sqrt{N} < t \leq N} \sum_q \sum_{q'} \tau H(q, t) \bar{\tau} H(q', t) = U_3 + U_4,$$

où U_3 est la contribution des termes avec $q = q'$, tandis que U_4 est la contribution des autres termes. Dans chaque terme de $\sum_q$ on a

$q > (\log t)^\omega \equiv (\frac{1}{2}n)^\omega$; en appliquant (32), où m est remplacé par le plus petit entier $\equiv \frac{2\omega}{\omega-m}$, on obtient donc

$$|U_3| \leq c_{53} N \sum_{q > (\frac{1}{2}n)^\omega} q^{-2 + \frac{\omega-m}{\omega}} < c_{54} N n^{-m}.$$

De la même manière que dans la démonstration de la proposition 2 pour le résultat analogue on trouve

$$\begin{aligned} |U_4| &\leq \left\{ \sum_{q=1}^{[n^\sigma]} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} q \left| \lambda\left(\frac{a}{q}\right) \lambda'\left(\frac{a}{q}\right) \right| \right\}^2 \\ &\equiv \gamma_1^4 \left\{ \sum_{q=1}^{[n^\sigma]} q^{2l+2} \right\}^2 \text{ en vertu de (3)} \\ &< c_{55} n^{2\sigma(2l+3)} < c_{56} N n^{-m}. \end{aligned}$$

De la même manière on démontre qu'à l'expression

$$\sum_{\sqrt{N} < t \leq N} \sum_q |\sum_q H(q, t)|^2 = \sum_{\sqrt{N} < t \leq N} \sum_q \sum_{q'} H(q, t) \overline{H}(q', t)$$

la contribution des termes avec $q = q'$ est inférieure à

$$c_{57} N \sum_{q > n^\sigma} q^{-2 + \frac{\sigma-m}{\sigma}} < c_{58} N n^{-m}$$

et que la contribution des autres termes est tout au plus égale à

$$\gamma_1^4 \left\{ \sum_{q=1}^{\left[N^{\frac{1}{4l+7}} \right]} q^{2l+2} \right\}^2 < c_{59} N^{\frac{4l+6}{4l+7}} < c_{60} N n^{-m}.$$

Pour tout $Q \equiv 2 N^{\frac{1}{4l+7}}$ nous avons $\log t \leq n < (4l+7) \log Q$; en appliquant de lemme 5 (p. 353) avec $\varrho = \frac{1}{2+2\omega}$ et $x = (\log t)^{1+\omega}$, nous trouvons que la somme $\sum_q$ renferme moins de

$$c_{61} Q^{1 - \frac{1}{2+2\omega}} e^{8(\omega+1)\sqrt{\log t}} < c_{62} Q^{1 - \frac{1}{3+3\omega}}$$

termes tels que l'on ait $\frac{1}{2}Q < q \leq Q$. La contribution de ces termes à $\sum_q H(q, t)$ est, en vertu de (32), où m est remplacé par le plus petit

entier $\equiv 6 + 6\omega$, inférieure à $c_{61} Q^{-\frac{1}{6+6\omega}}$. On a donc

$$\left| \sum_q H(q, t) \right| < c_{64} N^{-\frac{1}{(6+6\omega)(4l+7)}} < c_{65} n^{-m},$$

d'où suit l'inégalité (33) pour $\tau = 8$.

Ainsi nous avons obtenu (33) pour $\tau = 6, 7$ et 8 , de sorte que la proposition 5 est démontrée.

Cette proposition me permet de trouver au lieu de la proposition 4 le résultat suivant :

Proposition 6: *Si m est un nombre positif et si ν est supérieur à $3mg + 2m + 2g$, presque tout entier $t \equiv 4$ tel que $t - \psi(0)$ soit premier avec G satisfait à la relation*

$$F(t) = \left(1 + \frac{\theta}{(\log t)^m}\right) \phi(t) \Omega_\nu(t) \quad \text{où } |\theta| < 1.$$

Le nombre des exceptions $\equiv N$ est même pour tout $N \equiv 3$ inférieur à $c_{66} N (\log N)^{-m}$, où c_{66} dépend uniquement de m, ν et du choix du polynôme $\psi(x)$.

Dit d'une autre façon: on peut remplacer le second membre $2^{g-2} (3mg + 2m + 2g)$ de (30) par $3mg + 2m + 2g$. Il va sans dire que la proposition 4 suit immédiatement du dernier théorème pour tout $g \equiv 2$. Dans le cas particulier où g est égal à 1, la proposition 4 est évidente; en effet $F(t)$ est alors simplement le nombre des manières dont on peut écrire t sous la forme $p + bx + b'$, où b et b' sont des entiers donnés ($b > 0$); l'inégalité (31) est alors valable pour tout nombre naturel suffisamment grand t tel que $t - b'$ soit premier avec b .

Les propositions 3 et 6 ne sont que des cas particuliers des propositions 7 et 8 figurant dans la suivante application.

Deuxième application.

Introduisons un polynôme

$$\psi(x) = bx^g + \dots \quad (b > 0, g > 0)$$

du $g^{\text{ième}}$ degré qui prend des valeurs entières pour toutes les valeurs entières de x ; introduisons en outre deux nombres naturels K et U et un entier u premier avec U . Posons la question quels nombres naturels t possèdent la forme $Kp + \psi(x)$, où p est un nombre premier $\equiv u \pmod{U}$ et x un nombre naturel.

Désignons par G le plus grand commun diviseur de tous les nombres $\psi(x) - \psi(0)$, où x est entier. Soit E l'ensemble des nombres naturels t qui satisfont pour tout facteur premier P de KUG à la condition suivante, dans laquelle P^ω désigne la puissance la plus élevée de P qui est un diviseur de KU .

1. Si P n'est pas un facteur de U , il existe au moins un entier X tel que $\psi(X) - t$ soit divisible par P^ω , mais non par $P^{\omega+1}$.

2. Si P est un facteur de U , il existe au moins un entier X tel que $\psi(X) + Ku - t$ soit divisible par P^ω .

Considérons d'abord un nombre naturel t qui n'appartient pas à cet ensemble E et qui possède la susdite forme $Kp + \psi(x)$ où $p \equiv u \pmod{U}$.

Puisque t n'appartient pas à E , au moins un facteur premier P de KUG jouit de la propriété suivante:

1. Si P n'est pas un facteur de U , tout nombre de la forme $\psi(X) - t$ (où X est entier) qui est divisible par P^ω , est même divisible par $P^{\omega+1}$. Parce que U n'est pas divisible par P , le nombre K est divisible par P^ω , de sorte que $-Kp = \psi(x) - t$ est divisible par P^ω , c'est-à-dire par $P^{\omega+1}$, d'où il suit que p est divisible par P , donc égal à P .

2. Si P était un facteur de U , il n'y aurait aucun entier X tel que $\psi(X) + Ku - t$ soit divisible par P^ω ; ce cas est donc exclu, puisque $\psi(x) + Ku - t$ est divisible par KU , par conséquent par P^ω .

Ainsi nous avons obtenu le résultat suivant:

Si un nombre naturel t , n'appartenant pas à E , possède la susdite forme $Kp + \psi(x)$, où $p \equiv u \pmod{U}$, ce nombre p est nécessairement un facteur de KG .

Pour les nombres appartenant à E je déduirai le résultat suivant:

Proposition 7: *Presque tout entier appartenant à E possède la susdite forme. A tout nombre naturel m correspond même un nombre c_{67} dépendant uniquement de m, K, U et du choix du polynôme $\psi(x)$ tel que le nombre des exceptions $\equiv N$ soit pour tout $N \equiv 3$ inférieur à $c_{67} N (\log N)^{-m}$.*

Je vais traiter plus amplement un cas particulier. Je me demande quels nombres naturels t possèdent la forme $t = 3p + x^4$, où p est un nombre premier $\equiv 1 \pmod{3}$ et où x désigne un nombre naturel. Dans ce cas on a

$$K = U = 3, \quad u = G = 1, \quad \psi(x) = x^4;$$

E est l'ensemble des nombres naturels t auxquels correspond au moins un entier X tel que $X^4 + 3 - t$ soit divisible par 9. E est donc l'ensemble des nombres naturels t qui sont congrus à 1, 3, 4 ou 7 $\pmod{9}$. Ainsi nous obtenons le résultat suivant: presque tout nombre naturel t qui est congru à 1, 3, 4 ou 7 $\pmod{9}$ possède la forme $t = 3p + x^4$, où p est un nombre premier $\equiv 1 \pmod{3}$. De la même manière nous trouvons: presque tout nombre naturel t qui est congru à 1, 4, 6 ou 7 $\pmod{9}$ peut être mis sous la forme $t = 3p + x^4$, où le nombre premier p est $\equiv 2 \pmod{3}$.

Afin de démontrer la proposition 7, il suffit de traiter le cas particulier où $\psi(x)$ est un polynôme à coefficients entiers. En effet, supposons que ce cas particulier est déjà démontré. Les coefficients du polynôme $g! \psi(x)$ sont entiers. $g!G$ est le plus grand commun diviseur de tous les nombres $g! \psi(x) - g! \psi(0)$, où x est entier. Soit E^* l'ensemble des nombres naturels τ qui satisfont pour tout facteur premier P de $g!g!KUG$ à la condition suivante, dans laquelle P^z désigne la puissance la plus élevée de P qui est un diviseur de $g!KU$:

1. Si P n'est pas un facteur de U , il existe au moins un entier X tel que $g! \psi(X) - \tau$ soit divisible par P^z , mais non par P^{z+1} .

2. Si P est un facteur de U , il existe au moins un entier X tel que $g! \psi(X) + g! Ku - t$ soit divisible par p^z .

Afin de démontrer que pour tout élément t de E le produit $g! t$ appartient à E^* , je distingue trois cas différents:

I. Soit P un facteur de KG , mais pas de U . L'entier X tel que $\psi(X) - t$ soit divisible par P^ω , mais non par $P^{\omega+1}$, satisfait à la condition que $g! \psi(X) - g! t$ est divisible par P^z , mais non par P^{z+1} .

II. Soit P un facteur de U . L'entier X tel que $\psi(X) + Ku - t$ soit divisible par P^ω satisfait à la condition que $g! \psi(X) + g! Ku - g! t$ est divisible par P^z .

III. Soit P un facteur de $g!$, mais pas de KUG . Alors P^z est la puissance la plus élevée de P qui est un diviseur de $g!$. Puisque P n'est pas un facteur du plus grand commun diviseur G de tous les nombres $\psi(x) - \psi(0)$, où x est entier, on peut trouver un entier X tel que $\psi(X) - t$ ne soit pas divisible par P , de sorte que $g! \psi(X) - g! t$ est divisible par P^z , mais non par P^{z+1} .

Ainsi j'ai démontré que $g! t$ appartient à E^* pour tout nombre t appartenant à E . D'après la proposition 7, appliquée avec $g! \psi(x)$, $g! K$, $g! G$, $g! N$ et E^* au lieu de $\psi(x)$, K , G , N et E , presque tout nombre $g! t$ appartenant à E^* possède la forme $g! Kp + g! \psi(x)$, dans laquelle le nombre premier p est congru à $u \pmod{U}$, tandis que x est un nombre naturel. Le nombre des exceptions $\equiv g! N$ est pour tout $N \equiv 3$ inférieur à $c_{52} N (\log N)^{-m}$, où c_{52} dépend uniquement de m , K , U et du choix du polynôme $\psi(x)$. Par conséquent presque tout entier appartenant à E possède la forme $Kp + \psi(x)$, où p est congru à $u \pmod{U}$ et le nombre des exceptions $\equiv N$ est pareillement inférieur à $c_{52} N (\log N)^{-m}$.

Dans ce qui suit dans cette deuxième application je me bornerai à un polynôme $\psi(x)$ dont tous les coefficients sont entiers.

Désignons par $F(t)$ le nombre des manières dont on peut écrire un entier donné $t > 1$ sous la susdite forme; posons

$$\phi(t) = K^{-1} g^{-1} b^{-\frac{1}{g}} \cdot \varphi^{-1}(U) \sum_{h=2}^{t-2K} \frac{h^{-1+\frac{1}{g}}}{\log(t-h) - \log K} \cdot \dots \quad (34)$$

$(\phi(t))$ s'annule donc pour tout $t < 2K + 2$ et

$$\Omega_v^*(t) = \prod_{p \leq (\log t)^v} W(p, t),$$

où $W(p, t)$ est défini pour tout entier t et pour tout nombre premier p de la manière suivante (dans cette définition p^ω est la puissance la plus élevée de p , qui est un diviseur de KU):

pour un nombre premier p qui n'est pas un facteur de U , le produit $(p-1) W(p, t)$ est le nombre des nombres naturels $h \equiv p^{\omega+1}$ tels que $\psi(h) - t$ soit divisible par p^ω , mais non par $p^{\omega+1}$;

pour un facteur premier p de U le nombre $W(p, t)$ est égal au nombre des nombres naturels $h \leq p^u$ tels que $\psi(h) + Ku - t$ soit divisible par p^u .

$W(p, t)$ est positif pour tout entier $t > 1$ et pour tout nombre premier p qui n'est pas un facteur de KUG . En effet, $(p-1)W(p, t)$ est alors le nombre des nombres naturels $h \leq p$ tels que $\psi(h) - t$ ne soit pas divisible par p ; la relation $W(p, t) = 0$ impliquerait que p est un facteur de tous les nombres $\psi(x) - t$ (x entier), par conséquent de $\psi(x) - \psi(0)$, de sorte que p serait un facteur de G .

Pour un facteur premier p de KUG il suit de la définition de l'ensemble E que $W(p, t)$ est positif ou nul, selon que l'entier $t > 1$ appartient à E ou non. Si l'on choisit l'entier $t > 1$ et le nombre v tels que $(\log t)^v$ soit supérieur ou égal au plus grand facteur premier de KUG , le nombre $\Omega_v^*(t)$ est donc positif ou nul, selon que t appartient à E ou non.

Comme nous le verrons, $F(t)$ possède pour presque tout entier $t > 1$ la valeur approximative $\phi(t)\Omega_v^*(t)$, où v est assez grand. Cette remarque est évidente pour les entiers $t > 1$ n'appartenant pas à E . En effet, pour un tel nombre, $F(t)$ est, comme nous l'avons observé, borné, tandis que $\phi(t)\Omega_v^*(t)$ s'annule si l'on choisit v supérieur ou égal au plus grand facteur premier de KUG . Pour les nombres t appartenant à E la remarque résulte de la proposition suivante:

Proposition 8: *Si m est positif et si v est supérieur à $3mg + 2m + 2g$, presque tout nombre t appartenant à E satisfait à la relation*

$$F(t) = \left(1 + \frac{\theta}{(\log t)^m}\right) \phi(t) \Omega_v^*(t) \text{ où } |\theta| < 1.$$

Le nombre des exceptions $\leq N$ est même pour tout $N \geq 3$ inférieur à $c_{69} N(\log N)^{-m}$, où c_{69} dépend uniquement de m, v, K, U et du choix du polynôme $\psi(x)$.

Il va sans dire que la proposition 7 est une conséquence immédiate de la proposition 8. Cette dernière proposition résulte de la proposition suivante, comme je le démontrerai dans la suite.

Proposition 9: *Si M est un nombre positif et si v est supérieur à gM , on a pour tout nombre $N \geq 3$*

$$\sum_{t=2}^{[N]} |F(t) - \phi(t) \Omega_v^*(t)|^2 < c_{70} N^{1+\frac{2}{g}} n^{-M},$$

où c_{70} dépend uniquement de M, v, K, U et du choix du polynôme $\psi(x)$.

Pour la démonstration de cette proposition j'ai besoin de 7 lemmes.

Lemme 7: (Théorème de SIEGEL-WALFISZ). *A tout nombre naturel m correspond un nombre c_{71} , dépendant uniquement de m , tel que l'inégalité*

$$\left| \sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} 1 - \frac{1}{\phi(k)} \int_2^x \frac{du}{\log u} \right| < \frac{c_{71} x}{(\log x)^m}$$

soit vérifiée pour toute valeur $\equiv 2$ de x , pour tout nombre naturel k et pour tout entier h qui est premier avec k .

Lemme 8: A toute paire de nombres naturels m et K correspond un nombre c_{72} dépendant uniquement de m et de K tel que l'inégalité

$$\left| \sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} 1 - \frac{1}{K \varphi(k)} \sum_{2 \leq v \leq Kx} \frac{1}{\log \frac{v}{K}} \right| < \frac{c_{72} x}{(\log x)^m}$$

soit vérifiée pour toute valeur $\equiv 2$ de x , pour tout nombre naturel k et pour tout entier h qui est premier avec k .

Démonstration: Ce lemme est une conséquence immédiate du lemme précédent, puisqu'on a

$$\begin{aligned} \left| \int_2^x \frac{du}{\log u} - \frac{1}{K} \sum_{2 \leq v \leq Kx} \frac{1}{\log \frac{v}{K}} \right| &= \frac{1}{K} \left| \int_{2K}^{Kx} \frac{du}{\log \frac{u}{K}} - \sum_{2 \leq v \leq Kx} \frac{1}{\log \frac{v}{K}} \right| \\ &< c_{73} + \frac{1}{K} \sum_{2K \leq v \leq Kx} \int_v^{v+1} du \left(\frac{1}{\log \frac{v}{K}} - \frac{1}{\log \frac{u}{K}} \right) \\ &\equiv c_{73} + \frac{1}{K} \sum_{2K \leq v \leq Kx} \left(\frac{1}{\log \frac{v}{K}} - \frac{1}{\log \frac{v+1}{K}} \right) \\ &< c_{73} + \frac{1}{K \log 2}; \end{aligned}$$

c_{73} dépend uniquement de K .

Lemme 9: A tout polynôme

$$\psi(x) = bx^g + \dots \quad (b > 0)$$

du degré $g \equiv 1$ à coefficients entiers correspond un nombre positif c_{74} jouissant de la propriété suivante:

Si nous désignons par A' un nombre positif, par B' un nombre $> A'$, par y un nombre réel, par q et h des nombres naturels et si Σ^* est étendu à tous les entiers $v' > A'$ et $\equiv B'$ auxquels correspond au moins un nombre naturel $x \equiv h \pmod{q}$ tel que $\psi(x) = v'$, nous avons

$$\left| \sum_{v' \leq y} \Sigma^* 1 - q^{-1} g^{-1} b^{-\frac{1}{g}} \sum_{\substack{A' < v' \leq B' \\ v' \leq y}} v'^{-1 + \frac{1}{g}} \right| < c_{74}.$$

Démonstration: Dans le cas où y est $\equiv A'$ le premier membre s'annule. Dans le cas où $A' < y \equiv B'$ ce premier membre ne change pas sa valeur si l'on remplace B' par y . On peut donc supposer $y \equiv B'$.

Sans troubler la généralité on peut admettre en outre $A' \equiv c_{75}$, où cette constante, dépendant uniquement du choix du polynôme $\psi(x)$, est choisie si grande que $\psi(v')$ est une fonction croissante de v' pour $v' \equiv c_{75}$. A chaque point v' de l'intervalle $A' \equiv v' \equiv B'$ correspond alors un seul nombre $x \equiv c_{75}$ tel que $\psi(x)$ soit égal à v' . En choisissant $\alpha \equiv c_{75}$ et $\beta \equiv c_{75}$ de telle façon que nous ayons $\psi(\alpha) = A'$ et $\psi(\beta) = B'$, nous obtenons

$$\sum_{v'}^* 1 = \sum_{\substack{\alpha < x \leq \beta \\ x \equiv h \pmod{q}}} 1,$$

par conséquent

$$\left| \sum_{v'}^* 1 - \frac{\beta - \alpha}{q} \right| \leq 1 \dots \dots \dots (35)$$

En vertu de

$$B' = \psi(\beta) = b\beta^g + \dots$$

on a

$$b\beta^g = B' \left\{ 1 + O\left(B'^{-\frac{1}{g}}\right) \right\},$$

par conséquent

$$\beta = \left(\frac{B'}{b}\right)^{\frac{1}{g}} \left\{ 1 + O\left(B'^{-\frac{1}{g}}\right) \right\} = \left(\frac{B'}{b}\right)^{\frac{1}{g}} + O(1)$$

et on obtient d'une manière analogue

$$\alpha = \left(\frac{A'}{b}\right)^{\frac{1}{g}} + O(1),$$

La relation

$$g^{-1} b^{-\frac{1}{g}} \sum_{A' < v' \leq B'} v'^{-1 + \frac{1}{g}} = \left(\frac{B'}{b}\right)^{\frac{1}{g}} - \left(\frac{A'}{b}\right)^{\frac{1}{g}} + O(1)$$

nous apprend donc

$$\left| \beta - \alpha - g^{-1} b^{-\frac{1}{g}} \sum_{A' < v' \leq B'} v'^{-1 + \frac{1}{g}} \right| < c_{76},$$

c_{76} dépendant uniquement du choix du polynôme $\psi(x)$, de sorte que l'inégalité qui est à démontrer est une conséquence de (35).

Lemme 10: A toute paire de nombres naturels g et m correspondent un nombre naturel h et un nombre positif c_{77} jouissant de la propriété suivante :

Pour tout entier P , tout entier $Z \equiv 3$, et tout nombre réel β tels que

l'intervalle $(\beta \mp Z^{-g} (\log Z)^h)$ ne contienne aucune fraction à dénominateur positif $\equiv (\log Z)^h$ et pour tout polynôme réel ¹⁾

$$\chi(x) = \frac{x^\beta}{g!} + \dots$$

du $g^{\text{ième}}$ degré, on a

$$\left| \sum_{x=P+1}^{P+Z} e^{2\pi i \beta \chi(x)} \right| < c_{77} Z (\log Z)^{-m}$$

On peut trouver la démonstration dans mon article: Ueber Summen von Primzahlen und Primzahlquadraten, qui paraîtra bientôt dans les Mathematische Annalen (§ 3; lemme 3).

Lemme 11: A tout système de nombres naturels g, m et b correspondent un nombre naturel ζ et un nombre positif c_{78} jouissant de la propriété suivante:

Pour tout entier P , tout entier $Z \geq 3$ et tout nombre réel α tels que l'intervalle $(\alpha \mp Z^{-g} (\log Z)^\zeta)$ ne contienne aucune fraction à dénominateur positif $\equiv (\log Z)^\zeta$, et pour tout polynôme réel

$$\psi(x) = b x^g + \dots$$

du $g^{\text{ième}}$ degré, on a

$$\left| \sum_{x=P+1}^{P+Z} e^{2\pi i \alpha \psi(x)} \right| < c_{78} Z (\log Z)^{-m}.$$

Démonstration: On peut supposer $\log Z \geq g! b$. Sinon le lemme est évident.

Je considère les nombres h et c_{77} indiqués dans le lemme précédent et je pose $\zeta = h + 1$. Supposons que l'intervalle $(\alpha \mp Z^{-g} (\log Z)^\zeta)$ ne contienne aucune fraction à dénominateur positif $\equiv (\log Z)^\zeta$. L'intervalle $(g! b \alpha \mp Z^{-g} (\log Z)^h)$ ne contient aucune fraction $\frac{a}{q}$ à dénominateur positif $\equiv (\log Z)^h$; en effet, sinon la fraction $\frac{a}{g! b q}$, dont le dénominateur est

$$g! b q \equiv g! b (\log Z)^h \equiv (\log^\zeta Z)^\zeta,$$

serait situé dans l'intervalle $(\alpha \mp \frac{1}{g! b} Z^{-g} (\log Z)^h)$, donc dans l'intervalle $(\alpha \mp Z^{-g} (\log Z)^\zeta)$. Si l'on pose

$$\psi(x) = g! b \chi(x) \quad \text{et} \quad \beta = g! b \alpha$$

on obtient donc moyennant le lemme précédent

$$\left| \sum_{x=P+1}^{P+Z} e^{2\pi i \alpha \psi(x)} \right| = \left| \sum_{x=P+1}^{P+Z} e^{2\pi i \beta \chi(x)} \right| < c_{77} Z (\log Z)^{-m}.$$

¹⁾ Je désigne par $(\alpha \mp \eta)$ l'intervalle fermé $(\alpha - \eta, \alpha + \eta)$.

Lemme 12: Si γ , γ' et γ'' sont positifs et si la fonction réelle $Z(p)$ est pour tout nombre premier $p \equiv \gamma'$ tout au plus égal à $\frac{\gamma}{p-1}$ et pour tout nombre premier $< \gamma'$ tout au plus égal à γ'' , on a pour tout $\chi \equiv 2$

$$\prod_{p \leq \chi} |1 + Z(p)| \leq c_{79} (\log \chi)^\gamma,$$

où c_{79} dépend uniquement de γ , γ' et γ'' .

Si l'on sait en outre pour tout nombre premier p

$$Z(p) \equiv -1 + \frac{1}{\gamma''}, \dots \dots \dots (36)$$

on a

$$\prod_{p \leq \chi} (1 + Z(p)) \equiv c_{80} (\log \chi)^\gamma,$$

c_{80} désignant un nombre positif, dépendant uniquement de γ , γ' et γ'' .

Démonstration. Désignons par γ_0 le plus grand des deux nombres $2\gamma + 2$ et γ' .

Pour tout nombre δ qui est en valeur absolue $\leq \frac{1}{2}$, le nombre

$$\log(1 + \delta) = \delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} - \dots$$

est d'une part $\equiv \delta$ et d'autre part

$$\equiv \delta - \frac{1}{2} \delta^2 \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots \right) = \delta - \delta^2,$$

par conséquent

$$e^{\delta - \delta^2} \equiv 1 + \delta \equiv e^\delta.$$

Pour tout nombre premier $p \equiv \gamma_0$ on a donc

$$e^{-\frac{\gamma}{p-1} - \frac{\gamma^2}{(p-1)^2}} \equiv 1 - \frac{\gamma}{p-1} \equiv 1 + Z(p) \equiv 1 + \frac{\gamma}{p-1} \equiv e^{\frac{\gamma}{p-1}},$$

de sorte que

$$\prod_{\gamma_0 \leq p \leq \chi} (1 + Z(p))$$

est d'une part

$$\equiv e^{\sum_{p \leq \chi} \frac{\gamma}{p-1}} \equiv e^{\gamma (\log \log \chi + c_{81})} = c_{82} (\log \chi)^\gamma$$

et d'autre part

$$\equiv e^{-\sum_{p \leq \chi} \left(\frac{\gamma}{p-1} + \frac{\gamma^2}{(p-1)^2} \right)} \equiv e^{-\gamma (\log \log \chi + c_{83})} = c_{84} (\log \chi)^\gamma,$$

où c_{81} , c_{82} , c_{83} et c_{84} désignent des nombres positifs, dépendant uniquement de γ .

En outre on a

$$\prod_{p < \gamma_0} |1 + Z(p)| < c_{85}$$

et si l'on peut utiliser (36)

$$\prod_{p < \gamma_0} (1 + Z(p)) > c_{86},$$

où c_{85} et c_{86} sont nombres positifs, dépendant uniquement de γ_0 et γ'' .
Ainsi le lemme est démontré.

(Communicated at the meeting of April 30, 1938).

§ 20. Die Gleichung $M_{xy}^{uv} F_{uv}^{xy} = 0$.

Wir haben jetzt alle Gleichungen zu ermitteln, die mit konstanten Koeffizienten zwischen den F_{uv}^{xy} möglich sind. Machen wir also mit konstanten M_{xy}^{uv} den Ansatz

$$M_{xy}^{uv} F_{uv}^{xy} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179)$$

und nehmen wir für F die Gestalt (124) oder, ausführlicher geschrieben:

$$F_{uv}^{xy} = \hat{B}_{\varrho u, v} d^{\varrho xy} - \frac{5}{24} \hat{B}_{u\varrho, \tau} d^{y\varrho\tau} (v'x) - \frac{1}{24} \hat{B}_{v\varrho, \tau} d^{x\varrho\tau} (u'y) + \frac{1}{4} \hat{B}_{u\varrho, \tau} d^{x\varrho\tau} (v'y) - \left\{ \begin{aligned} & - \hat{C}^{\varrho x, y} d_{\varrho uv} + \frac{5}{24} \hat{C}^{x\varrho, \tau} d_{v\varrho\tau} (u'y) + \frac{1}{24} \hat{C}^{y\varrho, \tau} d_{u\varrho\tau} (v'x) - \frac{1}{4} \hat{C}^{x\varrho, \tau} d_{u\varrho\tau} (v'y), \end{aligned} \right\} \quad (180)$$

dann wird aus (179)

$$\left\{ \begin{aligned} & \hat{B}_{\varrho u, v} d^{\varrho xy} M_{xy}^{uv} - \frac{5}{24} \hat{B}_{u\varrho, \tau} d^{y\varrho\tau} M_{\lambda y}^{u\lambda} - \frac{1}{24} \hat{B}_{v\varrho, \tau} d^{x\varrho\tau} M_{x\lambda}^{\lambda v} + \frac{1}{4} \hat{B}_{u\varrho, \tau} d^{x\varrho\tau} M_{x\lambda}^{u\lambda} - \\ & - \hat{C}^{\varrho x, y} d_{\varrho uv} M_{xy}^{uv} + \frac{5}{24} \hat{C}^{x\varrho, \tau} d_{v\varrho\tau} M_{x\lambda}^{\lambda v} + \frac{1}{24} \hat{C}^{y\varrho, \tau} d_{u\varrho\tau} M_{\lambda y}^{u\lambda} - \frac{1}{4} \hat{C}^{x\varrho, \tau} d_{u\varrho\tau} M_{x\lambda}^{u\lambda} = \end{aligned} \right\} \quad (181)$$

$$= \hat{B}_{\varrho\tau, \tau} N^{\varrho\tau\tau} - \hat{C}^{\varrho\sigma, \tau} N_{\varrho\sigma\tau} = 0,$$

wobei bei den M mit einem λ unten und oben über λ zu summieren ist.

Dies muss Null sein in allen unabhängigen $\hat{B}$ und $\hat{C}$, d.h. wir haben:

$$\hat{B}_{\varrho\tau, \tau} N^{\varrho\tau, \tau} = 0 \quad \text{und} \quad \hat{C}^{\varrho\sigma, \tau} N_{\varrho\sigma\tau} = 0. \quad . \quad . \quad . \quad . \quad (182)$$

Was die erste dieser Gleichungen betrifft: wir haben in den §§ 6 und 8 nachgewiesen, dass ausser der Symmetrie $\hat{B}_{\varrho\tau, \tau} = \hat{B}_{\tau\varrho, \tau}$ und der zyklischen Symmetrie

$$\hat{B}_{\varrho\sigma, \tau} + \hat{B}_{\sigma\tau, \varrho} + \hat{B}_{\tau\varrho, \sigma} = 0$$

keine Abhängigkeiten zwischen den $\hat{B}$ bestehen. (Ich verbessere bei dieser Gelegenheit Gleichung (72), bei der zwei Terme fehlen. Sie hat zu lauten

$$\overline{M}^{1,34} - \overline{M}^{1,43} = \overline{N}_{2,56} - \overline{N}_{2,65}.$$

Hiernach sind die darauf folgenden Gleichungen zu ergänzen. Das Resultat,

dass zwischen den B und C keine linearen Abhängigkeiten bestehen, bleibt dasselbe). Daraus folgt, dass die Koeffizienten $N^{\varrho\sigma\tau}$ in (182) für alle Indizes ϱ, σ, τ die Gleichungen befriedigen

$$N^{\varrho\sigma\tau} + N^{\sigma\varrho\tau} - N^{\tau\varrho\sigma} - N^{\varrho\tau\sigma} = 0 \quad . \quad . \quad . \quad . \quad (183)$$

Une dasselbe gilt für die $N_{\varrho\sigma\tau}$.

Nach (181) haben wir

$$\left. \begin{aligned} N^{\varrho\sigma\tau} &= d^{\varrho xy} M_{xy}^{\sigma\tau} - \frac{5}{24} d^{x\sigma\tau} M_{\lambda x}^{\varrho\lambda} - \frac{1}{24} d^{x\sigma\tau} M_{x\lambda}^{\lambda\varrho} + \frac{1}{4} d^{x\sigma\tau} M_{x\lambda}^{\varrho\lambda} \\ N_{\varrho\sigma\tau} &= d_{\varrho uv} M_{\sigma\tau}^{uv} - \frac{5}{24} d_{u\sigma\tau} M_{\varrho\lambda}^{\lambda u} - \frac{1}{24} d_{u\sigma\tau} M_{\lambda\varrho}^{u\lambda} + \frac{1}{4} d_{u\sigma\tau} M_{\varrho\lambda}^{u\lambda} \end{aligned} \right\} \quad (184)$$

$N^{\varrho\sigma\tau}$ ist eine Linearform der d^{ijk} , die man wie folgt schreiben kann:

$$N^{\varrho\sigma\tau} = d^{xyz} [\delta_z^{\varrho} M_{xy}^{\sigma\tau} - \frac{5}{24} \delta_y^{\sigma} \delta_z^{\tau} M_{\lambda x}^{\varrho\lambda} - \frac{1}{24} \delta_y^{\sigma} \delta_z^{\tau} M_{x\lambda}^{\lambda\varrho} + \frac{1}{4} \delta_y^{\sigma} \delta_z^{\tau} M_{x\lambda}^{\varrho\lambda}] \quad (185)$$

und analog für $N_{\varrho\sigma\tau}$.

Setzen wir (185) in (183) ein, so erscheint d^{xyz} multipliziert mit

$$\left. \begin{aligned} &\delta_z^{\varrho} M_{xy}^{\sigma\tau} + \delta_z^{\sigma} M_{xy}^{\tau\varrho} - \delta_z^{\tau} M_{xy}^{\varrho\sigma} - \delta_z^{\varrho} M_{xy}^{\tau\sigma} \\ &- \frac{5}{24} [\delta_y^{\sigma} \delta_z^{\tau} M_{\lambda x}^{\varrho\lambda} + \delta_y^{\varrho} \delta_z^{\tau} M_{\lambda x}^{\sigma\lambda} - \delta_y^{\varrho} \delta_z^{\sigma} M_{\lambda x}^{\tau\lambda} - \delta_y^{\sigma} \delta_z^{\tau} M_{\lambda x}^{\varrho\lambda}] - \\ &- \frac{1}{24} [\delta_y^{\sigma} \delta_z^{\tau} M_{x\lambda}^{\lambda\varrho} + \delta_y^{\varrho} \delta_z^{\tau} M_{x\lambda}^{\lambda\sigma} - \delta_y^{\varrho} \delta_z^{\sigma} M_{x\lambda}^{\tau\lambda} - \delta_y^{\sigma} \delta_z^{\tau} M_{x\lambda}^{\lambda\varrho}] + \\ &+ \frac{1}{4} [\delta_y^{\sigma} \delta_z^{\tau} M_{x\lambda}^{\varrho\lambda} + \delta_y^{\varrho} \delta_z^{\tau} M_{x\lambda}^{\sigma\lambda} - \delta_y^{\varrho} \delta_z^{\sigma} M_{x\lambda}^{\tau\lambda} - \delta_y^{\sigma} \delta_z^{\tau} M_{x\lambda}^{\varrho\lambda}] \end{aligned} \right\} \quad . \quad . \quad (\varphi) \quad . \quad (186)$$

Hier haben wir über xyz zu alternieren und dies gleich Null zu setzen, woraus sich notwendige Bedingungsgleichungen für die M ergeben. Es sind dann fünf verschiedene Fälle zu unterscheiden, wenn wir den Fall bei Seite lassen, dass xyz und $\varrho\sigma\tau$ zwei verschiedene Indextripel sind und demnach in (φ) alles zu Null wird. Die fünf genannten Fälle sind:

Fall 1: x, y, z, σ, τ verschieden und $\varrho = x$. Aus (φ) folgt dann

$$M_{yz}^{\sigma\tau} - M_{yz}^{\tau\sigma} - M_{yz}^{\tau\varrho} + M_{yz}^{\varrho\sigma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (\varphi_1)$$

Fall 2: x, y, z, ϱ, τ verschieden, $\sigma = x$; aus (φ) folgt

$$M_{yz}^{\varrho\tau} - M_{yz}^{\tau\varrho} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\varphi_2)$$

Bei vier verschiedenen Indizes ist also M_{xy}^{uv} bezüglich x und y symmetrisch; (φ_1) wird dann erfüllt und wird überflüssig. Dual zu (φ_2) erhält man aus $N_{\varrho\sigma\tau}$ auf dieselbe Weise

$$M_{yz}^{\varrho\tau} - M_{yz}^{\tau\varrho} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\varphi'_2)$$

sodass wir also auch Symmetrie in den oberen Indizes haben.

Fall 3: x, y, z, σ verschieden. Hier drei Unterfälle:

$$3a) \varrho = x, \tau = x; \quad 3b) \varrho = x, \tau = y; \quad 3c) \varrho = \sigma, \tau = x.$$

3a) gibt wegen (φ_2) keine neue Gleichung. Bei 3b) erhält man dagegen:

$$M_{yz}^{\sigma y} - M_{zy}^{\sigma y} - M_{yz}^{\gamma \sigma} + M_{zy}^{\gamma \sigma} - M_{xz}^{\sigma \gamma} + M_{xz}^{\sigma \gamma} - \frac{5}{24} M_{\lambda z}^{\sigma \lambda} - \frac{1}{24} M_{z\lambda}^{\sigma \lambda} + \frac{1}{4} M_{z\lambda}^{\sigma \lambda} = 0. \quad (\varphi_3)$$

und dual hierzu, ausgehend von $N_{\varrho \sigma \tau}$:

$$M_{zy}^{\sigma \gamma} - M_{zy}^{\sigma \gamma} - M_{yz}^{\gamma \sigma} + M_{yz}^{\gamma \sigma} - M_{xz}^{\sigma \gamma} + M_{xz}^{\sigma \gamma} - \frac{5}{24} M_{\lambda z}^{\sigma \lambda} - \frac{1}{24} M_{z\lambda}^{\sigma \lambda} + \frac{1}{4} M_{z\lambda}^{\sigma \lambda} = 0. \quad (\varphi'_3)$$

Durch Subtraktion erhält man

$$M_{xz}^{\sigma \gamma} - M_{xz}^{\sigma \gamma} = \frac{1}{6} (M_{\lambda z}^{\sigma \lambda} - M_{z\lambda}^{\sigma \lambda}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\varphi''_3)$$

Wir bemerken, dass hier und im Folgenden über zwei gleiche lateinische Indizes nicht summiert wird. Dagegen ist bei $M_{\lambda z}^{\sigma \lambda}$ u.s.w. über λ zu summieren.

Schliesslich gibt der Fall 3c:

$$M_{yz}^{uu} = M_{zy}^{uu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\varphi_4)$$

und dual hierzu

$$M_{xx}^{uv} = M_{xx}^{uv} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\varphi'_4)$$

Fall 4: x, y, z, ϱ verschieden und $\sigma = x, \tau = y$. (Die Annahme $\sigma = \varrho, \tau = x$ führt auf den Fall 3c zurück). Aus (186) finden wir:

$$M_{yz}^{\varrho y} - M_{zy}^{\varrho y} - M_{xz}^{\varrho x} + M_{xz}^{\varrho x} - \frac{5}{12} M_{\lambda z}^{\varrho \lambda} - \frac{1}{12} M_{z\lambda}^{\varrho \lambda} + \frac{1}{2} M_{z\lambda}^{\varrho \lambda} = 0 \quad . \quad (\varphi_5)$$

Dual hierzu erhält man kein neues (φ'_5) , da $(\varphi_5) - (\varphi'_5) = 0$ wegen (φ'_3) erfüllt ist.

Fall 5: x, y, z verschieden. Hier zwei Unterfälle:

$$5a) \varrho = x, \sigma = y, \tau = z \quad \text{und} \quad 5b) \varrho = x, \sigma = y, \tau = x.$$

Bei 5a) ergibt sich aus (186):

$$\left. \begin{aligned} M_{yz}^{yz} - M_{zy}^{yz} - M_{yz}^{zy} + M_{zy}^{zy} + M_{xz}^{xz} - M_{xz}^{xz} - M_{xy}^{xy} + M_{yx}^{xy} - \\ - \frac{5}{24} [2 M_{\lambda x}^{x\lambda} - M_{\lambda y}^{y\lambda} - M_{\lambda z}^{z\lambda}] - \\ - \frac{1}{24} [2 M_{x\lambda}^{\lambda x} - M_{y\lambda}^{\lambda y} - M_{z\lambda}^{\lambda z}] + \\ + \frac{1}{4} [2 M_{x\lambda}^{x\lambda} - M_{y\lambda}^{y\lambda} - M_{z\lambda}^{z\lambda}] = 0. \end{aligned} \right\} \quad . \quad . \quad . \quad (\varphi_6)$$

Und schliesslich erhält man im letzten Falle 5b):

$$M_{yz}^{yx} - M_{zy}^{yx} - 2 M_{yz}^{xy} + 2 M_{zy}^{xy} + M_{xz}^{xx} - M_{xz}^{xx} + \frac{5}{8} M_{\lambda z}^{x\lambda} + \frac{1}{8} M_{z\lambda}^{\lambda x} - \frac{3}{4} M_{z\lambda}^{x\lambda} = 0 \quad (\varphi_7)$$

Dual zu (φ_6) bzw. (φ_7) haben wir

$$\left. \begin{aligned} M_{yz}^{yz} - M_{zy}^{yz} - M_{yz}^{zy} + M_{zy}^{zy} + M_{xz}^{xz} - M_{xz}^{xz} - M_{xy}^{xy} + M_{yx}^{xy} - \\ - \frac{5}{24} [2 M_{x\lambda}^{\lambda x} - M_{y\lambda}^{\lambda y} - M_{z\lambda}^{\lambda z}] - \\ - \frac{1}{24} [2 M_{\lambda x}^{x\lambda} - M_{\lambda y}^{y\lambda} - M_{\lambda z}^{z\lambda}] + \\ + \frac{1}{4} [2 M_{x\lambda}^{x\lambda} - M_{y\lambda}^{y\lambda} - M_{z\lambda}^{z\lambda}] = 0 \end{aligned} \right\} \quad . \quad . \quad . \quad (\varphi'_6)$$

und

$$M_{yz}^{yx} - M_{yz}^{xy} - 2M_{zy}^{yx} + 2M_{zy}^{xy} + M_{zz}^{xx} - M_{zz}^{xx} + \frac{5}{8}M_{z\lambda}^{\lambda x} + \frac{1}{8}M_{\lambda z}^{x\lambda} - \frac{3}{4}M_{z\lambda}^{x\lambda} = 0 \quad (\varphi_7')$$

Wir bilden noch $(\varphi_6) - (\varphi_6')$ und erhalten

$$M_{yx}^{xy} - M_{xy}^{yx} = - (M_{zx}^{xx} - M_{xz}^{xx}) + \left. \begin{aligned} &+ \frac{1}{6} [2(M_{\lambda x}^{x\lambda} - M_{x\lambda}^{\lambda x}) - (M_{\lambda y}^{y\lambda} - M_{y\lambda}^{\lambda y}) - (M_{z\lambda}^{\lambda z} - M_{\lambda z}^{z\lambda})] \end{aligned} \right\} \cdot (\varphi_6'')$$

Hier sind x, y und z untereinander verschieden. Analog ergibt sich aus $(\varphi_7) - (\varphi_7')$:

$$M_{zy}^{yx} - M_{yz}^{xy} + (M_{zx}^{xx} - M_{xz}^{xx}) - (M_{zz}^{xx} - M_{zz}^{xx}) = \frac{1}{2}(M_{z\lambda}^{\lambda x} - M_{\lambda z}^{x\lambda}) \quad \cdot \cdot \quad (\varphi_7'')$$

§ 21. Diskussion der Gleichungen (φ_i) .

Summieren wir in (φ_3) über den Index y , so entsteht wegen

$$M_{\lambda z}^{\tau\lambda} = \sum_{y \neq \tau, z} M_{yz}^{\tau y} + M_{\tau z}^{\tau\tau} + M_{zz}^{\tau z} :$$

$$M_{xz}^{x\tau} - M_{zx}^{\tau x} = -\frac{1}{24}M_{\lambda z}^{\tau\lambda} - \frac{5}{24}M_{z\lambda}^{\lambda\tau} + \frac{1}{4}M_{\lambda z}^{\lambda\sigma} \quad \cdot \cdot \cdot \quad (187)$$

Summieren wir hier noch über x , so kommt

$$M_{z\tau}^{\tau\tau} - M_{\tau z}^{\tau\tau} = -\frac{1}{6}M_{\lambda z}^{\tau\lambda} + \frac{1}{6}M_{z\lambda}^{\lambda\tau} \quad \cdot \cdot \cdot \quad (188)$$

Setzen wir (187) in (φ_3) ein, so erhalten wir

$$M_{xz}^{\tau x} - M_{zx}^{x\tau} = \frac{5}{24}M_{\lambda z}^{\tau\lambda} + \frac{1}{24}M_{z\lambda}^{\lambda\tau} - \frac{1}{4}M_{z\lambda}^{\lambda\sigma} \quad \cdot \cdot \cdot \quad (189)$$

Durch diese Gleichung wird auch (φ_5) überflüssig.

Aus (φ_7) wird jetzt nach den letzten drei Gleichungen:

$$M_{\lambda z}^{\lambda\tau} = M_{z\lambda}^{\tau\lambda}, \quad \cdot \cdot \cdot \quad (190)$$

während (φ_7'') nach (188) und der damit dualen Gleichung

$$M_{zz}^{\tau z} - M_{zz}^{z\tau} = \frac{1}{6}(M_{\lambda z}^{\tau\lambda} - M_{z\lambda}^{\lambda\sigma})$$

übergeht in (φ_3'') :

$$M_{xz}^{\tau x} - M_{zx}^{x\tau} = \frac{1}{6}(M_{\lambda z}^{\tau\lambda} - M_{z\lambda}^{\lambda\sigma}). \quad \cdot \cdot \cdot \quad (191)$$

Die Gleichungen (187) bis (191) treten jetzt an die Stelle von (φ_3) , (φ_5) , (φ_7) und den hierzu dualen. Man kann jetzt die Grössen $M_{xz}^{x\tau}$, $M_{zx}^{\tau x}$ und $M_{zz}^{x\tau}$ durch $M_{xz}^{\tau x}$ und die einmal-verjüngten M 's ausdrücken und erhält:

$$M_{xz}^{x\tau} = M_{xz}^{\tau x} - \frac{5}{24}M_{\lambda z}^{\tau\lambda} - \frac{1}{24}M_{z\lambda}^{\lambda\tau} + \frac{1}{4}M_{\lambda z}^{\lambda\sigma} \quad \cdot \cdot \cdot \quad (187a)$$

$$M_{zx}^{\tau x} = M_{xz}^{\tau x} - \frac{5}{24}M_{\lambda z}^{\tau\lambda} - \frac{1}{24}M_{z\lambda}^{\lambda\tau} + \frac{1}{4}M_{\lambda z}^{\lambda\sigma} \quad \cdot \cdot \cdot \quad (189a)$$

$$M_{zz}^{x\tau} = M_{xz}^{\tau x} - \frac{1}{6}M_{\lambda z}^{\tau\lambda} + \frac{1}{6}M_{z\lambda}^{\lambda\sigma} \quad \cdot \cdot \cdot \quad (191a)$$

Hier sind x , σ und z verschiedene Indizes. Bei $x=z$ hat man nach (190)

$$M_{zz}^{z\sigma} = M_{zz}^{\sigma z} - \frac{1}{6} M_{\lambda z}^{\sigma\lambda} + \frac{1}{6} M_{z\lambda}^{\lambda\sigma} \quad . \quad . \quad . \quad (190a)$$

und bei $x=\sigma$ gibt (188):

$$M_{z\sigma}^{\sigma\sigma} = M_{\sigma z}^{\sigma\sigma} - \frac{1}{6} M_{\lambda z}^{\sigma\lambda} + \frac{1}{6} M_{z\lambda}^{\lambda\sigma} \quad . \quad . \quad . \quad (188a)$$

Es bleiben jetzt noch die Gleichungen (φ_6) und (φ'_6) auf dieselbe Art auszuwerten. Wir setzen zur Abkürzung (über λ und μ wird addiert):

$$M = M_{\lambda\mu}^{\lambda\mu}, \quad M' = M_{\mu\lambda}^{\lambda\mu}.$$

Wenn wir dann in (φ_6) über alle vier $y \neq x, z$ addieren, so entsteht

$$\left. \begin{aligned} & 4(M_{xx}^{xz} - M_{xz}^{xz}) + M_{xz}^{zx} - M_{zx}^{zx} = \\ & = \frac{2}{24} M_{\lambda x}^{x\lambda} + \frac{9}{24} M_{x\lambda}^{\lambda x} - \frac{9}{24} M_{x\lambda}^{x\lambda} + \frac{9}{24} M_{\lambda z}^{z\lambda} + \\ & + \frac{9}{24} M_{z\lambda}^{\lambda z} - \frac{6}{24} M_{z\lambda}^{z\lambda} - M_{\lambda z}^{\lambda z} + \frac{1}{4} (M - M'). \end{aligned} \right\} \quad . \quad . \quad (192)$$

Hier verjüngen wir nach z und bekommen

$$M_{x\lambda}^{x\lambda} = M_{\lambda x}^{\lambda x} \quad . \quad . \quad . \quad . \quad (193)$$

In (192) vertauschen wir x mit z und subtrahieren; wegen (193) gibt dies

$$-M_{xz}^{xz} + M_{zx}^{xz} - M_{xz}^{zx} + M_{zx}^{zx} = \frac{1}{6} (M_{\lambda x}^{x\lambda} - M_{x\lambda}^{\lambda x}) - \frac{1}{6} (M_{\lambda z}^{z\lambda} - M_{z\lambda}^{\lambda z}). \quad (194)$$

Wenn wir von (192) die duale Gleichung subtrahieren, so erhalten wir, (was auch aus (φ'_6) abzuleiten):

$$M_{xz}^{xz} - M_{zx}^{zx} = \frac{1}{6} (M_{\lambda x}^{x\lambda} - M_{x\lambda}^{\lambda x}) - \frac{1}{6} (M_{\lambda z}^{z\lambda} - M_{z\lambda}^{\lambda z}) \quad . \quad . \quad (195)$$

und dies gibt mit (194) zusammen:

$$M_{xz}^{xz} = M_{zx}^{zx}, \quad . \quad . \quad . \quad . \quad (193a)$$

wodurch (193) überflüssig wird.

Vertauscht man in (192) x mit z und addiert, so entsteht wegen (193a)

$$\left. \begin{aligned} & M_{xx}^{xz} + M_{zz}^{xz} - 2M_{xz}^{xz} = \\ & = \frac{1}{4} (M_{\lambda x}^{x\lambda} + M_{x\lambda}^{\lambda x}) - \frac{1}{2} M_{x\lambda}^{x\lambda} + \frac{1}{4} (M_{\lambda z}^{z\lambda} + M_{z\lambda}^{\lambda z}) - \frac{1}{2} M_{z\lambda}^{z\lambda} + \frac{1}{10} (M - M') \end{aligned} \right\} \quad (194a)$$

und aus dieser Gleichung und (195) findet man:

$$\left. \begin{aligned} & M_{xx}^{xz} - M_{zz}^{xz} = \\ & = \frac{5}{24} M_{\lambda x}^{x\lambda} + \frac{1}{24} M_{x\lambda}^{\lambda x} - \frac{1}{4} M_{x\lambda}^{x\lambda} + \frac{1}{24} M_{\lambda z}^{z\lambda} + \frac{5}{24} M_{z\lambda}^{\lambda z} - \frac{1}{4} M_{z\lambda}^{z\lambda} + \\ & + \frac{1}{20} (M - M'). \end{aligned} \right\} \quad (194b)$$

Jetzt kann man die ersten acht Terme von (φ_6) mit Hilfe von (194a) und (194b) ausdrücken. Aus (φ_6) entsteht dann

$$-\frac{1}{24} M_{\lambda x}^{x\lambda} - \frac{2}{24} M_{x\lambda}^{\lambda x} + \frac{1}{2} M_{x\lambda}^{x\lambda} - \frac{2}{24} M_{\lambda z}^{z\lambda} - \frac{1}{24} M_{z\lambda}^{\lambda z} + \frac{1}{2} M_{z\lambda}^{z\lambda} = \frac{1}{10} (M - M'). \quad (196)$$

Hieraus durch Verjüngung mit z :

$$5 M_{\lambda x}^{x\lambda} + M_{x\lambda}^{\lambda x} = 6 M_{x\lambda}^{x\lambda};$$

dies gibt im Vereine mit der hierzu dualen Gleichung:

$$M_{\lambda x}^{x\lambda} = M_{x\lambda}^{\lambda x} = M_{x\lambda}^{x\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (197)$$

und also nach (196):

$$M = M'$$

und nach (194b) im Vereine mit (193a) schliesslich

$$M_{xx}^{xx} = M_{xx}^{xx} = M_{xx}^{xx} = M_{xx}^{xx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (198)$$

wodurch (φ_6) und (φ'_6) erfüllt sind.

§ 22. Die irreduziblen Syzygien dritter Art.

Wir kehren zurück zum Ansatz (179) oder

$$M_{xy}^{uv} F_{uv}^{xy} \equiv 0 \{ \hat{B}, \hat{C} \} \quad (\text{über alle vier Indizes wird summiert}). \quad (199)$$

Sind zunächst alle vier Indizes x, y, u und v untereinander verschieden, so lassen sich in (199) nach (φ_2) und (φ'_2) je vier Glieder zusammenfassen zu

$$M_{xy}^{uv} (F_{uv}^{xy} + F_{vu}^{xy} + F_{uv}^{yx} + F_{vu}^{yx}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

wobei jetzt über die gleichen Indizes nicht summiert wird.

Nach (129a) verschwindet dies identisch. Es bleiben also in (199) nur noch solche Terme stehen, bei denen die vier Indizes von M nicht alle untereinander verschieden sind. Dies gibt, wenn wir mit $x = y = u = v$ beginnen, zunächst die Terme

$$M_{xx}^{xx} F_{xx}^{xx}.$$

Sie fallen weg wegen der aus (129) folgenden Gleichung

$$F_{xx}^{xx} \equiv 0 \{ \hat{B}, \hat{C} \}.$$

Dasselbe gilt für die Glieder mit F_{xx}^{uu} .

Bei drei verschiedenen Indizes x, y, z haben wir nach (187a) bis (191a)

$$\begin{aligned} M_{xz}^{yx} F_{yx}^{xz} &= M_{xz}^{yx} F_{yx}^{xz} \\ M_{xz}^{xy} F_{xy}^{xz} &= M_{xz}^{yx} F_{xy}^{xz} + M_z^y F_{xy}^{xz} \\ M_{zx}^{yx} F_{yx}^{zx} &= M_{zx}^{yx} F_{yx}^{zx} + M_z^y F_{yx}^{zx} \\ M_{zx}^{xy} F_{xy}^{zx} &= M_{zx}^{yx} F_{xy}^{zx} - N_z^y F_{xy}^{zx} \end{aligned} \quad \text{wobei} \quad \begin{aligned} M_z^y &= \frac{1}{4} M_{\lambda z}^{\lambda y} - \frac{5}{24} M_{\lambda z}^{y\lambda} - \frac{1}{24} M_{z\lambda}^{\lambda y} \\ N_z^y &= \frac{1}{6} (M_{\lambda z}^{y\lambda} - M_{z\lambda}^{\lambda y}). \end{aligned}$$

Diese Gleichungen addieren wir. Links entsteht eine Summe von vier

Termen aus (199), die wir kurz S_4 nennen; rechts gibt die Summe der ersten Terme so wie bei (200) Null. Es bleibt also

$$S_4 = M_z^y (F_{xy}^{xz} + F_{yx}^{zx}) - N_z^y F_{xy}^{zx} (201)$$

Dies summieren wir über alle vier Indizes x , die von y und z verschieden sind. Links bekommen wir eine Summe S_{16} von 16 Termen aus (199); rechts haben wir z.B.

$$\sum_{x \neq y, z} F_{xy}^{xz} = F_{\lambda y}^{\lambda z} - F_{yy}^{yz} - F_{zy}^{zz} .$$

Also wird:

$$S_{16} = M_z^y (F_{\lambda y}^{\lambda z} + F_{y\lambda}^{z\lambda}) - M_z^y (F_{yy}^{yz} + F_{zy}^{zz} + F_{yy}^{zy} + F_{yz}^{zz}) - \\ - N_z^y F_{\lambda y}^{z\lambda} + N_z^y (F_{yy}^{xy} + F_{zy}^{zz}) .$$

Der erste Term rechts ist Null nach (126), der zweite nach (129) und der dritte nach (127). Daher bleibt

$$S_{16} = N_z^y (F_{yy}^{zy} + F_{zy}^{zz}) (202)$$

Zu diesen 16 Termen fügen wir weitere vier, nämlich die mit

$$M_{yz}^{yy}, M_{zz}^{yz}, M_{zz}^{zy} \text{ und } M_{zy}^{yy},$$

wobei wir für die beiden letzten nach (190a) und (188a) schreiben können:

$$M_{zz}^{zy} F_{zy}^{zz} = M_{zz}^{yz} F_{zy}^{zz} - N_z^y F_{zy}^{zz} \\ M_{zy}^{yy} F_{yy}^{zy} = M_{yz}^{yy} F_{yy}^{zy} - N_z^y F_{yy}^{zy} .$$

Die Summe S'_4 dieser vier Terme wird dann, wieder nach (129):

$$S'_4 = -N_z^y (F_{zy}^{zz} + F_{yy}^{zy}) .$$

Also fällt auch $S_{16} + S'_4$ weg.

Es bleiben schliesslich in (199) noch die Terme mit zwei verschiedenen Indizes mit den Koeffizienten

$$M_{xz}^{xz}, M_{zx}^{xz}, M_{xz}^{zx}, M_{zx}^{zx} .$$

Auch hier können wir vermöge der Gleichheiten (198) wieder vier Glieder zusammenfassen zu

$$M_{xz}^{xz} (F_{xz}^{xz} + F_{zx}^{xz} + F_{xz}^{zx} + F_{zx}^{zx}) ,$$

was wieder nach (129a) Null wird.

Damit ist bewiesen, dass zwischen den F_{uv}^{xy} in der Tat keine anderen linearen Beziehungen (mit konstanten Koeffizienten) bestehen als

$$F_{uv}^{xy} + F_{vu}^{xy} + F_{uv}^{yx} + F_{vu}^{yx} = 0 (203)$$

$$F_{\lambda v}^{x\lambda} = 0, \quad F_{u\lambda}^{\lambda y} = 0 (204)$$

und die weiteren Gleichungen am Schlusse des § 14, insbes. die Beziehung

$$F_{\lambda v}^{\lambda y} + F_{v \lambda}^{y \lambda} = 0$$

sind eine Folge von (203).

Wir zählen jetzt noch die unabhängigen, also irreduziblen F_{uv}^{xy} ab. Im Ganzen gibt es $6^4 = 1296$ solche F_{uv}^{xy} . Davon fallen zufolge von (203) weg:

erstens die 36 möglichen $F_{uu}^{xx} = 0$, wobei auch $x = u$ sein kann; zweitens bei den

$$F_{uv}^{xx} \text{ mit } u \neq v \text{ und bei den } F_{uu}^{xy} \text{ mit } x \neq y$$

je 6.15, also 180 im Ganzen; drittens von den F_{uv}^{xy} mit $x \neq y$ und $u \neq v$ je eines, also $15.15 = 225$. Zuzufolge der Gleichungen (204) fallen schliesslich weitere 72 weg: also im Ganzen

$$36 + 180 + 225 + 72 = 513.$$

Somit gibt es genau $1296 - 513 = 783$ irreduzible Syzygien dritter Art.

Chemistry. — *Die akute Zinnpest II.* Von ERNST COHEN und W. A. T. COHEN—DE MEESTER.

(Communicated at the meeting of April 30, 1938).

Einleitung.

In unserer ersten, mit J. LANDSMAN veröffentlichten, Abhandlung ¹⁾ über die akute Zinnpest führten wir den Nachweis, dass die enorme Verzögerung, welche die Umwandlung des weissen Zinns in das graue häufig erfährt, dem Vorhandensein äusserst geringer Spuren gewisser Metalle (Bi, Pb, Sb) zuzuschreiben ist und dass der Zusatz äusserst geringer Mengen Aluminium (z.B. 0.001 bzw. 0.01 Gew. Proz.) zum deformierten, bzw. nicht-deformierten Zinn die U.G. desselben derart erhöht, dass eine „akute“ Zinnpest eintritt, welche in wenigen Stunden zur völligen Disgregation des weissen Zinns zu führen im Stande ist. Auch wurde auf die Bedeutung hingewiesen, welche diesen Erscheinungen für das Studium der Polymorphie anderer Metalle zukommt. Die in vorliegender Abhandlung beschriebene Untersuchung befasst sich mit der Lösung der Frage: Wie lässt sich die so stark beschleunigende Wirkung solch minimaler Aluminiumzusätze erklären?

1. *Wirkt mechanisch beigemengtes Aluminium beschleunigend?*

1. Zunächst war die Frage zu beantworten, ob die Umwandlung des weissen Zinns in das graue auch durch mechanisch beigemengtes Aluminium beschleunigt wird oder ob dies nur dann der Fall ist, wenn, wie in unseren früheren Untersuchungen, die beiden Metalle als Legierung vorliegen. Vorausgeschickt sei, dass sämtliche in vorliegender Abhandlung zu beschreibenden Versuche, wenn gegenteiliges nicht bemerkt wird, mit dem früher (Erste Mitteilung § 4) genannten spektroskopisch untersuchten, reinen Zinn von HILGER (H.Z.) ausgeführt wurden, welches 99.996 Proz. Zinn enthielt.

Das zu den betreffenden Versuchen verwendete Aluminiumpulver enthielt, wie die Untersuchung ergab, welche Herr Chem. Cand. J. CHR. DE WIJS für uns ausführte, als Verunreinigung 0.5 Proz. Aluminiumoxyd, sowie eine Spur Eisen.

2. Die Versuche gestalteten sich nunmehr wie folgt: Einen Stab

¹⁾ Proc. Royal Acad. Amsterdam, **40**, 746 (1937); Z. physik. Chem. (A) **181**, 124 (1937).

weisses H.Z. (Durchmesser 7 mm) walzten wir zu einem 0.9 mm dicken Streifen, entfetteten denselben durch waschen mit Aether und stellten uns aus dem Metall 20 Plättchen von je 3 cm Länge her. Zehn derselben brachten wir, jedes für sich, in ein Glasrohr, welches zugeschmolzen wurde. Mit den andern zehn Plättchen verfahren wir genau so, nachdem wir dieselben durch eintauchen in das Aluminiumpulver mit diesem Metall bedeckt hatten. Sodann brachten wir die 20 Röhrchen in unser Kältebad (-50°C.). Täglich untersuchten wir, ob die Zinnpest eingetreten war. Es ergab sich, dass das Aluminium unter den beschriebenen Verhältnissen einen Einfluss auf die U.G. nicht übt, m.a.W. die Beschleunigung tritt nur dann auf, wenn das Zinn mit dem Aluminium legiert ist.

Bei unseren weiteren Studien zur Erklärung des Einflusses, den Aluminium auf die U.G. des Zinns übt, haben wir uns auf Grund des soeben mitgeteilten, ausschliesslich den Zinn-Aluminium *Legierungen* zugewandt.

2. Der HEYN-WETZEL-Effekt.

3. Gelegentlich gewisser dilatometrischen Messungen am H.Z., welches mit 0.01 Gew. Proz. Aluminium (gleichfalls von HILGER, London bezogen, spektrographisch untersucht, Reinheitsgrad über 99.99 Proz.) legiert war, beobachteten wir, dass sich in dem Kapillarrohr (bei 25°C.) nach einiger Zeit Gasblasen zeigten. Als Dilatometerflüssigkeit war Heptan verwendet, welches während längerer Zeit mit P_2O_5 in Berührung gewesen war und das mit metallischem Natrium auch bei höherer Temperatur nicht die geringste Spur Wasserstoff entwickelte, somit völlig wasserfrei war. Da wir vermuteten, dass das sich im Dilatometer entwickelnde Gas Wasserstoff sei, welcher sich eventuell aus Wasser in Berührung mit der Zinnlegierung entwickelt haben könnte, untersuchten wir zunächst, ob bereits näheres über das Verhalten der Zinn-Aluminiumlegierungen Wasser, bezw. Wasserdampf gegenüber bekannt war. Wir fanden, dass E. HEYN und E. WETZEL¹⁾ vor längerer Zeit eine Untersuchung nach dieser Richtung ausführten aus Anlass der Tatsache, dass ein angeblich reines Zinn ungewöhnlich spröde war. Wir haben ihre Untersuchung, welche die Erklärung dieser Erscheinung bezweckte, wiederholt und konnten deren Ergebnisse an unseren Materialien in allen Punkten bestätigen.

4. Werden Zinnstäbe von 7 mm Durchmesser mit 0.05, 0.10, bezw. 0.25 Gew. Proz. Aluminium bei gewöhnlicher Temperatur zu Streifen von 0.2 mm heruntergewalzt, so sind dieselben unmittelbar nach dem Auswalzen biegsam und geschmeidig, wie reines Zinn. Sie lassen sich zwischen den Fingern wie Papier zusammenknittern, ohne brüchig oder

¹⁾ Z. Metallkunde, 14, 335 (1922).

rissig zu werden. Lässt man sie an der Luft liegen, so werden sie mit der Zeit in zunehmendem Masse spröde, und zwar schliesslich so, dass sie bei der geringsten Biegung wie Glas zerbrechen. Auf der Oberfläche der gewalzten Streifen entstehen nach kurzer Zeit zahlreiche metallische Aufblähungen, (Fig. 1), deren Wand die Farbe des weissen Zinns auf-

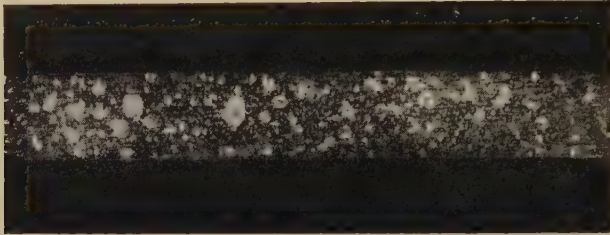


Fig. 1.

weist. Durchbricht man diese Wand mit einem Messer, so erblickt man im Innern eine weisse Substanz. (Vergl. weiter unten).

Bringt man die Metallstreifen sofort nach dem Walzen in einen mit Schwefelsäure beschickten Exsikkator, so bleiben die beschriebenen Erscheinungen aus: das Metall ist selbst nach monatelangem Liegen völlig biegsam und glänzend und auch die Warzenbildung ist nicht eingetreten. Setzt man das Material sodann der Einwirkung der (feuchten) Laboratoriumsluft aus, so treten die erörterten Erscheinungen ein. In Wasser, bezw. in einer einprozentigen Chlornatriumlösung lässt sich ein ungemein schneller Verlauf derselben beobachten.

5. Zusammenfassend, lässt sich feststellen, dass die hier erörterte Erscheinung, welche wir fernerhin mit dem Namen HEYN-WETZEL-effekt (H.W.E.) bezeichnen werden, in Folge der Zersetzung des Wassers durch die Zinn-Aluminiumlegierung eintritt, ein Vorgang, der bei aluminiumreichen Zinnlegierungen von W. CARRICK ANDERSON und G. LEAN¹⁾ eingehend studiert worden ist.

Es bildet sich beim H.W.E. Wasserstoff sowie Aluminiumhydroxyd. Letzteres lässt sich nachweisen, indem man die Metallstreifen nach dem Walzen in Wasser legt. Die Flüssigkeit trübt sich in Folge der Bildung der Hydroxyds, welches sich schliesslich zu Boden setzt. Die Entstehung der genannten metallischen Aufblähungen (Fig. 1) erklärt sich durch die Bildung des Hydroxyds in dem Metall. Dieselbe ist von einer bedeutenden Volumenzunahme (etwa 10 Proz.) begleitet, welche eine Dehnung des Metalls hervorruft, die schliesslich zum Reissen desselben und zum Austritt des Hydroxyds an die Metalloberfläche führt.

¹⁾ B. A. Rep., 1901, 606; Proc. Roy. Soc. London, 72, 277 (1903).

3. Die beschleunigende Wirkung des Aluminiums beim Auftreten der akuten Zinnpest.

6. Nach diesem Abstecher wenden wir uns wieder den oben (§ 3) beschriebenen Dilatometerversuchen zu.

Falls das dabei entwickelte Gas von einer Wasserzersetzung durch das aluminiumhaltige Zinn herrührt, war die Frage zu beantworten, woher dieses Wasser stammte, da das verwendete Heptan, wie oben betont wurde, völlig wasserfrei war.

Es lag auf der Hand zu vermuten, dass das hier eine Rolle spielende Wasser ¹⁾ der Legierung entstammte. Nun haben die Untersuchungen von TH. W. RICHARDS ²⁾ und seiner Mitarbeiter, sowie die von VINAL und BOVARD ³⁾ und von ERNST COHEN und H. R. BRUINS ⁴⁾ den Beweis erbracht, dass es schwer hält vollständig trockne Metalle darzustellen. Dieselben enthalten stets, selbst wenn eine Trocknung im Vakuumexsikkator über Schwefelsäure stattgefunden hat, geringe Mengen Wasser, deren Betrag von der Grössenordnung 0.01 bis 0.001 Gew. Proz. ist. Durch Schmelzen des Metalls in einem neutralen, trocknen Gasstrom lassen sich diese Spuren entfernen.

Unsere Zinn-Aluminiumlegierung war, bevor sie in das Dilatometer gegeben wurde, nicht allein mit der (nicht trocknen) Laboratoriumsluft in Berührung gewesen, sondern gleichfalls mit der Hand des Experimentators, als derselbe die längeren Metallstreifen in kleine Schnitzel zerteilte, um sie in das Dilatometer bringenzukönnen.

Zur näheren Kontrolle haben wir in 140 g Zinn die vorhandene Menge Wasser ermittelt und zwar nach dem von ERNST COHEN und H. R. BRUINS ⁴⁾ beschriebenen Verfahren. Wir fanden 7 mg Wasser (0.005 %), eine Menge, welche 8.5 cc Wasserstoff zu liefern im Stande ist.

7. Nach Feststellung dieser Tatsachen ergibt sich nunmehr folgende Erklärung für die enorm beschleunigende Wirkung geringer Aluminiummengen, bei der Umwandlung des weissen Zinns in die graue Modifikation.

Bringt man, wie es in unseren diesbezüglichen Versuchen (Erste Mitteilung) der Fall war, die gewalzte, bezw. gegossene Zinn-Aluminiumlegierung (in Glasröhren eingeschmolzen) in ein Kältebad (-50°C.), so tritt zunächst der HEYN-WETZELLEffekt ein. Das Metall bläht sich auf, was stellenweise zu einer Deformierung desselben Anlass gibt. Wie

¹⁾ Die in 50 g der 0.01 Gew. Proz. Al enthaltenden Legierung vorhandene Menge Aluminium wäre im Stande etwa 2.5 mg Wasser zu zersetzen und dabei etwa 3 cc Wasserstoff zu liefern.

²⁾ J. Am. Chem. Soc., **37**, 7, 675 (1915).

³⁾ J. Am. Chem. Soc., **38**, 515 (1916).

⁴⁾ Z. physik. Chem., **94**, 443 (1920).

früher¹⁾ nachgewiesen wurde, wirken diese Deformierungszentren als ebensoviele Keime, an welchen die Umwandlung des weissen Zinns in das graue einsetzt und von wo aus die Umwandlung dann infolge eintretender Impfung fortschreitet, d.h.: die U.G. des weissen Zinns wird durch die Bildung jener Zentren beschleunigt.

8. Ist diese Erklärung richtig, so liesse sich erwarten, dass die beschleunigende Wirkung des Aluminiums bei der Umwandlung des weissen Zinns in das graue nicht eintritt, falls die Legierung auch nicht die geringste Spur Wasser enthält, da dann ja der H.W.E. nicht auftreten kann. Dass solche Spuren bereits eine sehr ausgesprochene Wirkung zur Folge haben können, da sie ja hier zur Bildung von Keimen deformierten, weissen Zinns führen, ergibt sich u.a. aus der von Wl. OSTWALD²⁾ festgestellten Tatsache, dass zur Stabilisierung metastabiler Gebilde bereits Keime von der Grössenordnung 10^{-12} g genügen.

Wir versuchten somit eine völlig wasserfreie Zinn-Aluminiumlegierung (0.01 Gew. Proz. Al) herzustellen. Wie bereits oben (§ 6) betont wurde, gelingt dies nicht, wenn man ein festes Metall im Vakuum trocknet, wie auch in unserem Falle ein speziell dazu angestellter Versuch (Trockenmittel P_2O_5) ergab.

9. Sodann versuchten wir auch die letzten Spuren etwa vorhandenen Wassers folgenderweise zu entfernen: Die vorher äusserlich getrockneten Metalle wurden in einer Kugel aus Pyrexglas zusammengeschmolzen, die vorher bis nahe zum Erweichen des Glases erhitzt und an der Hochvakuumpumpe evakuiert war. Die geschmolzene Legierung wurde längere Zeit im Hochvakuum gehalten. Sodann schmolzen wir das Rohr zu und taten die flüssige Legierung in das 6 mm weite Rohr fliessen, das sich an der Glaskugel befand. Nach dem Erkalten zerschlugen wir das Rohr, schnitten den gebildeten Draht schnell in sechs Stücke, wobei jegliche Berührung mit der Hand vermieden wurde und gaben jedes Stück in ein zuvor hocheerhitztes Glasrohr, das evakuiert und unter den nötigen Kauteln zugeschmolzen wurde, um das Eintreten feuchter Luft zu vermeiden. Im Parallelversuch mit sechs Stücken der Legierung derselben Zusammensetzung, welche aber *nicht* unter Ausschluss der feuchten Laboratoriumsluft dargestellt waren, liess sich ein Unterschied in der U.G. bei -50°C . nicht nachweisen.

10. Nach diesem Ergebnis versuchten wir unsere oben gegebene Erklärung in anderer Weise zu erhärten und zwar indem wir den Vorgang

¹⁾ ERNST COHEN, W. A. T. COHEN-DE MEESTER und A. K. W. A. VAN LIESHOUT, Proc. Roy. Acad. Amsterdam, **38**, 377 (1935); ERNST COHEN und A. K. W. A. VAN LIESHOUT, Proc. Roy. Acad. Amsterdam, **39**, 352, 1174 (1936).

²⁾ Z. physik. Chem., **22**, 289 (1897).

der akuten Zinnpest in zwei Teile zerlegten, die, jeder für sich, der Beobachtung zugänglich sind. Da aber das Eintreten des H.W.E. sich bei dem 0.01 Gew. Proz. Al enthaltenden Zinn mit dem blossen Auge nicht wahrnehmen lässt (nur die Wasserstoffentwicklung im Dilatometer beweist, dass derselbe tatsächlich eintritt), verwendeten wir zu diesen Versuchen ein 0.25 Gew. Proz. Al enthaltendes Zinn, an welchem sich die den H.W.E. begleitenden Aufblähungen ohne weiteres beobachten lassen.

Längere Streifen (0.2 mm dick) dieses Materials liessen wir an der Luft liegen, bis der H.W.E. sichtbar geworden war: das Metall zeigte an mehreren Stellen die oben genannten, weissen, metallischen Aufblähungen. Sodann brachten wir dasselbe in Glasrohre, welche (nicht getrocknete) Laboratoriumsluft enthielten, schmolzen dieselben zu und brachten sie in das Kältebad (-50° C.). Bereits am nächsten Tage hatte sich die Zinnpest an den deformierten Stellen entwickelt.

11. Obwohl sich aus dem bisher mitgeteilten ergibt, dass der H.W.E. an sich zur akuten Zinnpest zu führen im stande ist, kann erst eine neue Untersuchung zeigen, ob auch andere Vorgänge dabei eine Rolle spielen. Wir hoffen hierüber demnächst zu berichten und dann auch die beschleunigende bzw. verlangsamende Wirking anderer Metalle in den Kreis unserer Betrachtungen zu ziehen.

Zusammenfassung.

In der hier beschriebenen Untersuchung wurde dargetan, dass bei der Erzeugung der akuten Zinnpest in Zinn, welches Spuren Aluminium enthält, der HEYN-WETZELeffekt eine Rolle spielt.

Utrecht, April 1938.

VAN 'T HOFF-Laboratorium.

Physics. — *A theory of plastic buckling with its application to geophysics* *). By P. P. BIJLAARD. (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of April 30, 1938).

As has now been sufficiently confirmed by tests and theoretical arguments, the yield-point will be reached in crystalline materials which become plastic when the stress is sufficiently high, if the specific shear energy attains a definite value. This plasticity condition, thought out by HUBER, VON MISES and HENCKY, for the case of plane states of stress with the principal stresses ϱ_1 and ϱ_2 , whilst $\varrho_3 = 0$, is expressed as follows:

$$\varrho_1^2 - \varrho_1 \varrho_2 + \varrho_2^2 = \sigma_v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where σ_v represents the yield-point for pure tension or compression.

The plastic deformations take place in what is called a quasi-isotropic manner, i.e. the stress-strain relation will be the same as for an isotropic elastic material, but the ordinary elasticity modulus E is replaced by a variable deformation modulus E_p , which generally speaking also depends on time, whilst the contraction coefficient m obtains the value 2, since the plastic deformations do not create any changes of volume ^{1, 2)}.

In addition it is assumed that the plastic deformation is at a given moment only determined by the then existing state of stress, in the same way as seems approximately to be the case with soft steel ³⁾.

If for all stress conditions $\varrho_1, \varrho_2, 0$ which satisfy eq. (1) we draw the MOHR circles determined by ϱ_1 and ϱ_2 , we find that they touch an enveloping ellipse, with the equation:

$$3\sigma^2 + 12\tau^2 - 4\sigma_v^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If this plasticity condition and deformation laws are applicable, it may be proved that in plates subjected to a plane state of stress $\varrho_1, \varrho_2, 0$ local plastic deformations, such as flow lines, c.q. necking, will occur, whenever the oblique stress $\mu = \sqrt{\sigma^2 + \tau^2}$ on the plane going through these flow lines c.q. this necking, has increased until it touches the envelope (2).

*) Most of the ideas brought forward in this article were mentioned already in a lecture delivered on 19th March 1938 in Amsterdam to the Committee for the Study of Viscosity of the Royal Academy of Sciences and the Geophysical Group of the Society of Geologists and Mining Engineers.

1) ROŠ. Eidg. Materialprüfungsanstalt E. T. H. Zürich. Diskussionsbericht, no. 34.

2) BIJLAARD. De Ingenieur, no. 23 (1933).

3) HOHENEMSER und PRAGER. Zeitschrift angew. Math. u. Mech., no. 1 (1932). PRAGER. Proceedings 3rd Congress for applied mechanics, Stockholm, Vol. II, p. 13 (1930).

This leads us to the conclusion that in a broad plate which is subjected to a plane state of stress the normal on the flow lines will form an angle α with the direction of ϱ_1 , for which 4):

$$\operatorname{tg} \alpha = \sqrt{(\varrho_1 - 2\varrho_2)/(2\varrho_1 - \varrho_2)} \quad (3)$$

If $\varrho_1 > \varrho_2$, then (3) will give imaginary values for $\varrho_2 > \frac{1}{2}\varrho_1$. For that particular case ϱ_1 will, in order to create a local plastic deformation, always have to increase to the value

$$2\sigma_v/\sqrt{3} = 1,154\sigma_v, \text{ while: } \alpha = 0 \quad (3a)$$

These laws may also be applied to the tangential stresses superposed on the hydrostatic pressure in the crust of the earth. The strong hydrostatic pressure permits the shearing-resistance of the crystallites to be exceeded without the (stray) tension stresses attaining the cohesion strength, and so the crust under a sufficient superposed plane state of stress $\varrho_1, \varrho_2, 0$ will deform plastically. As the deformation takes place slowly and at a high temperature, there will not be any work hardening, and so with pure compression the stress-strain diagram will be approximately such as shown by figure 1 5). The local plastic deformations in the crust of the earth which create the strips of negative gravitational anomalies of VENING

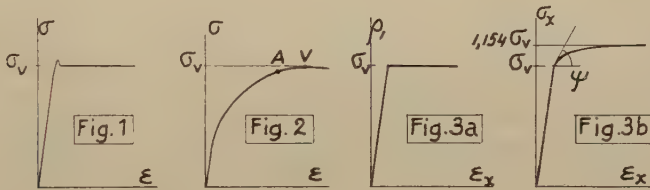


Fig. 1—3b.

MEINESZ, strings of islands and chains of mountains, will consequently also occur in the planes indicated by (3) or (3a). This leads us, amongst other things, to the conclusion that the strings of islands and the strips of negative anomalies in the Western Pacific were created by divergent compressional forces exerted by the Asiatic Continent, as well as to a plausible explanation of the origin of geosynclines, mountain chains and volcanism 6).

Although these things correspond satisfactorily to the available geological facts, it is still to be found out, whether the crust of the earth

4) BIJLAARD. De Ingenieur, no. 8 and 37 (1931). De Ingenieur in Ned.-Indië, no. 6 (1935), no. 8 (1936).

5) Further on compression stresses and shortenings are considered to be positive.

6) BIJLAARD. Int. Congress of the geodetic and geophysical Union, Edinburgh (1936). — 3rd Engineering Congress, Tokio (1936). — 7de Ned. Ind. Natuurwetenschappelijk Congres, Batavia (1935). — De Ingenieur in Ned.-Indië, no. 11 and 12 (1935); no. 4, 7 and 11 (1936).

cannot buckle before showing local plastic deformations. In the elastic domain the greatest principal stress ϱ_1 required for buckling is $2\sqrt{Nc}/h$, when $N = m^2 EI/(m^2 - 1)$ and $I = \frac{1}{12} h^3$. The modulus of elasticity for the crust of the earth may be $E = 840000 \text{ kg/cm}^2$, whilst $m = 5$ and the thickness h of the crust may be $3 \cdot 10^6 \text{ cm}$; if the crust lies under the surface of the water, the counterpressure c of the substratum amounts to $0,0023 \text{ kg/cm}^3$. It thus follows that $\varrho_1 = 45000 \text{ kg/cm}^2$. However, as far as experiments by ROŠ, ADAMS, BAILEY and others with crystalline materials under high pressure and at a high temperature appear to indicate, the material of the crust of the earth will begin to flow already under a superposed pure compression stress of about 3000 kg/cm^2 . Hence a possibility of buckling can eventually only exist in the plastic domain.

According to the now existing theory concerning the plastic buckling of plates, the so-called reduced modulus of elasticity T , whose value is between that of E and that of the total deformation modulus $E_t = d\sigma/d\varepsilon$ at the buckling stress, should be introduced instead of the ordinary elasticity modulus E . At the yield stress, where $E_t = d\sigma/d\varepsilon = 0$, $T = 0$ too⁷⁾. With a $\sigma - \varepsilon$ diagram as in figure 1, buckling below the yield-point would not be possible, but it would be with a $\sigma - \varepsilon$ diagram as in figure 2, which is improbable, however. In the latter diagram, in point A , E_t , for example, might become so small as to reduce T to less than $(3000/45000)^2 E = E/225$, which might cause the buckling stress ϱ_1 to become less than 3000 kg/cm^2 . Between A and V there would then be a chance of buckling before strips of local plastic deformation could have been formed at the yield-point V . The distance AV will, however, be comparatively small, and would therefore be rapidly covered, and in connection with the high viscosity of the substratum, there would be a good chance of V being reached before considerable buckling could have occurred.

As will be explained further on, the resistance to plastic buckling of a plate is, however, much greater than thus far has been admitted, and while at the yield-point the resistance of the plate to further compression indeed is zero, with the resistance to bending this is not the case. I once casually mentioned that the situation is more complicated than in the case of centric compression, because an other state of stress is superposed by the buckling on the original state of stress⁸⁾. Now this very fact appears to cause a rise of the yield stress, whenever the amount of bending is finite, as well as an increase of the resistance, whenever that amount is infinitely small.

If, for example, on an elastically supported plate, such as the crust of

⁷⁾ BLEICH, SCHLEICHER and ROŠ. Int. association for bridge and structural engineering. 1st Congress, Paris, Final Report, p. 120—149 (1932). — TIMOSHENKO. Theory of elastic stability, p. 386—390 (1936). — CHWALLA. Report of the 2nd int. congress for bridge and structural engineering, Vienna, p. 321—322 (1928).

⁸⁾ BIJLAARD. De Ingenieur, no. 26 (1934).

the earth, a pure compressional stress $\varrho_1 = \sigma_v$ is exercised in the X -direction, this plate may buckle in waves which are perpendicular to the direction of ϱ_1 . An element $dx dy dz$ in the concave side of a wave in case of continued buckling then will suffer a considerable shortening in the X -direction, whereas in the Y -direction, i.e. parallel to the direction of the waves, the actual bending will not result in a change of length⁹⁾. The material is then in the same condition as in a strip of local plastic deformation running in the Y -direction. As has been previously explained with a plate locally weakened in the Y -direction, but subjected to tension in the X -direction, considerable plastic deformations cannot occur in such a weakened strip, until σ_x has increased unto $2\sigma_v/\sqrt{3} = 1,154 \sigma_v$ ¹⁰⁾. The relation between σ_x and σ_y for the concave part of the wave may be calculated in the same way as was done before¹⁰⁾ for the weakened strip. The plastic strains are:

$$\varepsilon_{xp} = \frac{\sigma_x}{E_p} - \frac{\sigma_y}{2E_p} \quad (4a) \quad \text{and} \quad \varepsilon_{yp} = \frac{\sigma_y}{E_p} - \frac{\sigma_x}{2E_p} = \frac{2\sigma_y - \sigma_x}{2\sigma_x - \sigma_y} \varepsilon_{xp} \quad (4b)$$

The total strain in the Y -direction, including the elastic strain, is:

$$\varepsilon_y = \varepsilon_{ye} + \varepsilon_{yp} = \frac{\sigma_y}{E} - \frac{\sigma_x}{mE} + \frac{2\sigma_y - \sigma_x}{2\sigma_x - \sigma_y} \varepsilon_{xp} \quad (5)$$

so the total strain in the X -direction is:

$$\varepsilon_x = \varepsilon_{xe} + \varepsilon_{xp} = \frac{\sigma_x}{E} - \frac{\sigma_y}{mE} + \frac{2\sigma_x - \sigma_y}{2\sigma_y - \sigma_x} \left(\varepsilon_y + \frac{\sigma_x}{mE} - \frac{\sigma_y}{E} \right) \quad (6)$$

As σ_x and σ_y must continue to satisfy eq. (1) ($\varrho_1 = \sigma_x$, $\varrho_2 = \sigma_y$):

$$\sigma_y = \frac{1}{2} \sigma_x \pm \sqrt{\sigma_v^2 - \frac{3}{4} \sigma_x^2} \quad (7)$$

The minus sign in (7) applies to the original state of stress, hence this sign will now have to be taken into account. It follows from (6) and (7) that:

$$\varepsilon_x = \frac{(m-2)(3\sigma_x^2 - 2\sigma_v^2 + \sigma_x \sqrt{4\sigma_v^2 - 3\sigma_x^2}) - (3\sigma_x + \sqrt{4\sigma_v^2 - 3\sigma_x^2})mE\varepsilon_y}{2mE \sqrt{4\sigma_v^2 - 3\sigma_x^2}} \quad (8)$$

According to (8) $\varepsilon_x = \sigma_v/E$, with $\varepsilon_y = -\sigma_v/mE$, if $\sigma_x = \sigma_v$ and e.g. $\varepsilon_x = 5,64 \sigma_v/E$ if $\sigma_x = 1,15 \sigma_v$ ¹¹⁾, and thus, if the $\sigma - \varepsilon$ diagram for

⁹⁾ The X - and Y -axes are to be found in the median plane of the plate. The Z -direction falls perpendicularly to it. The original stresses in the X - and Y -directions are called ϱ_1 and ϱ_2 , the normal stresses which after bending are acting on the X and Y planes, and which generally will not be, strictly speaking, principal stresses (see further on), are called σ_x and σ_y .

¹⁰⁾ Cf. footnote 2.

¹¹⁾ The relation between σ_x and ε_x with finite bending may be used to find, at which maximum initial deflection $2f$ an originally locally bent strip which is isostatically

compression is of the type of figure 3a, fig. 3b will apply to bending. In case of centric buckling only infinitely slight bending is taken into consideration, hence also infinitely small strains will have to be considered. Consequently the yield-point is not enhanced here, but the angle ψ determining the slope of the $\sigma_x - \epsilon_x$ diagram, which expresses the stiffness with regard to bending, nevertheless has a great value at the yield-point. As, according to equation (8), $\sigma_x = 1,01\sigma_v$ if $\epsilon_x = 1,043\sigma_v/E$, this shows that $\tan \psi > E/4,3$. So far it has been admitted that in this case $\tan \psi$ was 0. It is easier than it would be by differentiation of equation (8), to determine $\tan \psi$, or the relation between stress and strain in buckling, in the following way.

By way of introduction we first admit $\varrho_1 = \sigma_v$, and $\varrho_2 = 0$, and we adopt an arbitrary form of buckling. On the X plane, on which acts ϱ_1 , extra twisting stresses τ'_{xy} will start working at buckling, accompanied by extra normal stresses σ'_x , as well as the extra stresses σ'_y and $\tau'_{yx} = \tau'_{xy}$ on the Y plane. The yield condition (1) now becomes:¹²⁾

$$\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = \sigma_v^2. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The infinitely small extra stresses which result from buckling, will either cause in the elements a discharge, leading back to the domain of completely elastic deformations, or they will not take away the plasticity, the point representative of the state of stress in that case having to remain always on the limiting ellipsoid (9). We will be able, for these infinitely small extra stresses, to replace the limiting surface by the tangential plane in the point A representative of the state of stress, which existed before buckling: $\sigma_x = \varrho_1 = \sigma_v$, $\sigma_y = \varrho_2 = 0$. The equation of this tangential plane,

compensated and runs in the Y direction, will just escape excentric buckling. If the average stress $\varrho_1 = \sigma_v$, and if we admit, for example, that the strain in the Y direction, through its being joined to the unbent crust is restricted to the strain occurring in the crust, then in equation (8) $\epsilon_y = -\sigma_v/mE$. At the convex side of the wave, where discharge occurs, the deformation is elastic. If the curvature is known, the internal moment may now be determined. Presuming the deflection to be sinusoidal, I could prove that the resulting maximum initial amplitude f is smallest, when the original deflection has a (half) wave length $a = \pi(h\varrho_1/2c)^{1/2}$. With this buckling length it is now possible to calculate the internal moment of the crust, as well as the resisting moment coming from the substratum. In the way shown by Roš at the 2nd Congress for applied mechanics at Zürich, p. 373 (1926) for excentrically loaded bars f may then be graphically determined. If, in connection with "creep" the calculation is made with a smaller E at the concave side, e.g. $E = 200000$ kg/cm², and in connection with thickening of the crust resulting from bending and sedimentation with a smaller c , e.g. $c = 0,0005$ kg/cm², then we find that $f = 1,10$ km. For a single strip the maximum initial deflection then becomes about $2f = 2,20$ km. We supposed σ_v to be 2600 kg/cm². If at the concave side we say that $E = 100000$ kg/cm², then $2f$ becomes 1,44 km.

¹²⁾ The normal and shearing stresses on the X and Y planes do their work independently of one another. If only shearing stresses τ_{xy} were at work, $\varrho_1 = -\varrho_2$ being τ_{xy} , then equation (1) would transform into $3\tau^2 = \sigma_v^2$. Thus when they act in combination with σ_x and σ_y equation (9) becomes the yield condition.

expressed in the extra stresses, which have their origin in the tangential point A , becomes:

$$2\sigma'_x - \sigma'_y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The extra twisting stresses τ'_{xy} apparently do not influence the combination of σ'_x and σ'_y which may occur at the yield-point¹³). If in (6), also in connection with (10), we put:

$$\sigma_x = \sigma_v + \sigma'_x, \sigma_y = \sigma'_y = 2\sigma'_x, \epsilon_x = \sigma_v/E + \epsilon'_x, \epsilon_y = -\sigma_v/mE + \epsilon'_y \quad (11)$$

ϵ'_x and ϵ'_y representing the strains created through buckling, then σ'_x , ϵ'_x and ϵ'_y will be the only unknown quantities in (6), hence σ'_x and consequently also σ'_y , may be expressed in ϵ'_x and ϵ'_y . We find:

$$\left. \begin{aligned} \sigma'_x &= \frac{mE}{5m-4} (\epsilon'_x + 2\epsilon'_y) \\ \sigma'_y &= \frac{2mE}{5m-4} (\epsilon'_x + 2\epsilon'_y) \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

As this is of particular importance for the calculation of the buckling stresses of steel plates used in bridge construction, the $\sigma' - \epsilon'$ relations will now be deducted for the most general case, whilst the $\tau_{xy} - \gamma_{xy}$ relation will be determined too. We assume that the plate buckles as the result of a state of stress ϱ_1, ϱ_2 , and put: $\varrho_2 = \beta\varrho_1$, β being < 1 . According to the plasticity condition (1) this state of stress is equivalent to a linear stress:

$$\sigma_q = \sqrt{\varrho_1^2 - \varrho_1\varrho_2 + \varrho_2^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

According to the Roš experiments¹⁴) with soft steel tubes, this plasticity condition also gives satisfactory results in the domain of strain hardening, where $d\sigma_q/d\epsilon = tg\varphi_t$ is not zero in case of unrestrained yield, meaning that the relation between σ_q and ϵ_q is the same for all states of stress, ϵ_q then being:

$$\epsilon_q = \sqrt{\epsilon_{xp}^2 + \epsilon_{xp}\epsilon_{yp} + \epsilon_{yp}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Roš deducted the relation between σ_q and ϵ_q by admitting the same E_p for all states of stress. If, for example, in (4a) and (4b) we replace σ_x and σ_y by ϱ_1 and ϱ_2 resp., combination with equation (13) will result in:

$$\sigma_q = \frac{2}{\sqrt{3}} E_p \sqrt{\epsilon_{xp}^2 + \epsilon_{xp}\epsilon_{yp} + \epsilon_{yp}^2} = \frac{2}{\sqrt{3}} E_p \epsilon_q \quad . \quad . \quad . \quad (15)$$

We generally assume that already with the linear stress σ_q there occur

¹³) Not the extra vertical shearing stresses τ'_{xz} and τ'_{yz} , of which the deformations are always ignored with buckling of full sections, either.

¹⁴) Cf. Note 1.

plastic deformations of amount $\varepsilon_p = \sigma_q/E_p = e\sigma_q/E$ (we put $E_p = E/e$) and that further $d\sigma_q/d\varepsilon_p = tg\varphi$ (fig 4b).

The point B representative of the state of stress ϱ_1, ϱ_2 is in analogy with (9) now situated on the ellipsoid:

$$\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = \sigma_q^2 \quad . \quad . \quad . \quad (16)$$

which intersects the σ_x axis in the point A representing the equivalent linear stress σ_q (fig. 4a)¹⁵.

The point C representing the state of stress resulting after buckling, if for the element examined σ_q increases through buckling, will be situated on a similar ellipsoid, which intersects the σ_x axis at a point A' , for which $\sigma_x = \sigma_q + d\sigma_q$ (fig. 4a). The two ellipsoids may in their turn

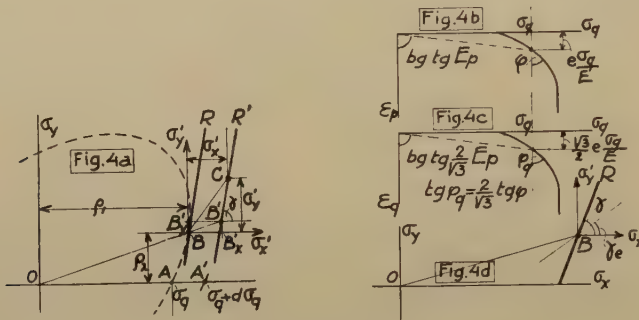


Fig. 4a—4d.

be replaced by the respective tangential planes R and R' in the respective points $\sigma_x = \varrho_1$, $\sigma_y = \varrho_2$, and $\sigma_x = \varrho_1 + \sigma'_x$, $\sigma_y = \varrho_2 + \sigma'_y$, which again are parallel to the τ_{xy} axis and form an angle γ with the σ_x axis, for which:

$$tg\gamma = \frac{\partial \sigma_y}{\partial \sigma_x} = \frac{2\sigma_x - \sigma_y}{\sigma_x - 2\sigma_y} = \frac{2\varrho_1 - \varrho_2}{\varrho_1 - 2\varrho_2} = \frac{2 - \beta}{1 - 2\beta} \quad . \quad . \quad . \quad (17)$$

Since the points A and A' , like B and B' , are corresponding points of the two ellipsoids, we have $BB'_x = \varrho_1 d\sigma_q/\sigma_q$ and $BB'_y = \varrho_2 d\sigma_q/\sigma_q$. The relation between σ'_x and σ'_y must be in accordance with figure 4a:

$$\sigma'_y = BB'_y + (\sigma'_x - BB'_x) tg\gamma = (\varrho_2 - \varrho_1 tg\gamma) d\sigma_q/\sigma_q + \sigma'_x tg\gamma \quad . \quad (18)$$

Hence in equation (6) we put:

$$\left. \begin{aligned} \sigma_x &= \varrho_1 + \sigma'_x \\ \sigma_y &= \varrho_2 + \sigma'_y = \varrho_2 + \sigma'_x tg\gamma + (\varrho_2 - \varrho_1 tg\gamma) d\sigma_q/\sigma_q \\ \varepsilon_x &= \varrho_1/E - \varrho_2/mE + e\varrho_1/E - e\varrho_2/2E + \varepsilon'_x \\ \varepsilon_y &= \varrho_2/E - \varrho_1/mE + e\varrho_2/E - e\varrho_1/2E + \varepsilon'_y \end{aligned} \right\} \quad . \quad . \quad . \quad (19)$$

¹⁵ In figure 4a the section of (16) with the plane $\tau_{xy} = 0$ is dotted on another scale.

According to figure 4c, (14) and (15), the intermediary transformations being omitted, we find:

$$\begin{aligned} d\sigma_q &= d\varepsilon_q \operatorname{tg} \varphi_q = \frac{2}{\sqrt{3}} \left(\frac{\partial \varepsilon_q}{\partial \varepsilon_{xp}} d\varepsilon_{xp} + \frac{\partial \varepsilon_q}{\partial \varepsilon_{yp}} d\varepsilon_{yp} \right) \operatorname{tg} \varphi = (\varrho_1 d\varepsilon_{xp} + \varrho_2 d\varepsilon_{yp}) \frac{\operatorname{tg} \varphi}{\sigma_q} = \\ &= \frac{(1-2\beta)mE \operatorname{tg} \varphi}{(1-2\beta)mE + (1-m\beta)2 \operatorname{tg} \varphi} \frac{\varrho_1}{\sigma_q} \left(\varepsilon'_x + \beta \varepsilon'_y - \frac{(m-2)(1-\beta^2)}{(1-2\beta)mE} \sigma'_x \right) \end{aligned} \quad (20)$$

After a number of transformations (6) gives σ'_x as a function of ε'_x and ε'_y , and then, with (18), we find σ'_y :

$$\sigma'_x = E(A\varepsilon'_x + B\varepsilon'_y), \quad \sigma'_y = E(C\varepsilon'_x + D\varepsilon'_y) \quad (21)$$

in which:

$$A = \varphi_1/\varphi_4, \quad B = \varphi_2/\varphi_4 = C, \quad D = \varphi_3/\varphi_4 \quad (22)$$

and:

$$\left. \begin{aligned} \varphi_1 &= m^2(1-2\beta) \{ (1-2\beta)^2 E + (4\eta^2 + 3e) \operatorname{tg} \varphi \} \\ \varphi_2 &= m(1-2\beta) \{ (2-\beta)(1-2\beta)mE + (4\eta^2 + 3em\beta) \operatorname{tg} \varphi \} \\ \varphi_3 &= m^2(1-2\beta) \{ (2-\beta)^2 E + (4\eta^2 + 3e\beta^2) \operatorname{tg} \varphi \} \\ \varphi_4 &= m(1-2\beta) \{ (5m-4)(1+\beta^2) + 2(5-4m)\beta + 3em\eta^2 \} E + \\ &\quad + [4(m^2-1)(1-2\beta)\eta^2 + 3em\{2(1-m\beta)\eta^2 + (m-2)(1-\beta^2)\}] \operatorname{tg} \varphi \\ \beta &= \varrho_2/\varrho_1 \quad \text{and} \quad \eta^2 = \sigma_q^2/\varrho_1^2 = \beta^2 - \beta + 1. \end{aligned} \right\} \quad (23)$$

The strains ε'_x and ε'_y are infinitely small, hence on buckling the finite stresses ϱ_1 and ϱ_2 produce infinitely small plastic deformations. The infinitely small twisting stresses τ'_{xy} consequently produce plastic shearing strains which are infinitely small of the second order. At the previous plastic deformation E_p was equal to E/e ; hence the plastic modulus of rigidity becomes $G_p = mE_p/2(m+1) = E/3e$. Thus:

$$\gamma'_{xy} = \tau'_{xy}/G + \tau'_{xy}/G_p = \{2(m+1)/m + 3e\} \tau'_{xy}/E = (2m+2+3em) \tau'_{xy}/mE$$

or

$$\tau'_{xy} = \frac{mE}{2m+2+3em} \gamma'_{xy} = EF\gamma'_{xy} \quad (24)$$

With regard to the crust of the earth the case $\operatorname{tg} \varphi = 0$ is of importance. It follows from (22) and (23) that then:

$$B/A = D/C = C/A = D/B = \sigma'_y/\sigma'_x = (2-\beta)/(1-2\beta) = \operatorname{tg} \gamma \quad (25)$$

This is to be expected, since the point representing the state of stress must remain on the tangential plane to the same ellipsoid which forms

the angle γ shown in equation (17) with the σ_x axis. In the elastic domain we know that:

$$\sigma'_x = \frac{m^2 E}{m^2 - 1} \left(\epsilon'_x + \frac{1}{m} \epsilon'_y \right) \text{ and } \sigma'_y = \frac{m^2 E}{m^2 - 1} \left(\frac{1}{m} \epsilon'_x + \epsilon'_y \right) \quad (26)$$

as follows likewise from equation (21), in which now $tg\varphi = \infty$ and $e = 0$; hence

$$\sigma'_y / \sigma'_x = (\epsilon'_x + m \epsilon'_y) / (m \epsilon'_x + \epsilon'_y) = tg \gamma_e \quad (27)$$

If we think of an arbitrary deformation of the plate at buckling, then the elements, the representative points of which for elastic deformation are situated to the left of the tangential plane R (fig. 4d), will be deformed elastically, where as those where this point would be found on the right of R would be deformed plastically, the representative point in that case remaining on R . As with bending the sections may be assumed to remain plane, ϵ'_x and ϵ'_y will vary linearly with z along the entire thickness of the plate, and so, according to (21) and (26), σ'_x and σ'_y for a certain small element $h dx dy$, will also vary linearly with z , both in the plastic and the elastic domain (fig. 5a). It should be noted, however, that equilibrium is only possible, if the plate deforms plastically along the entire height¹⁶⁾, and that then in

¹⁶⁾ The transition between the plastic and the elastic domain would have to be at a height h_p under the top of the plate (fig. 5a), where $tg \gamma_e = tg \gamma$ (fig. 4d), or where $\epsilon'_x = \epsilon'_y = 0$. The equilibrium requires $\iint \sigma'_x dh dy$ and $\iint \sigma'_y dh dx$ to be 0. As in the plastic domain σ'_y / σ'_x is constant, we note that, if $\iint \sigma'_x dh dy = 0$, $\iint \sigma'_y dh dx$ can only be 0, if also in the elastic domain σ'_y / σ'_x has the same constant ratio, and $tg \gamma_e = tg \gamma$, which as a rule will not be the case. In order to satisfy both conditions, the section will have to deform either altogether elastically, or altogether plastically. Entirely elastic deformation is not possible, for if the representative point of an element under the centre plane were to the left of R (fig. 4d), that of an element above the centre plane would be

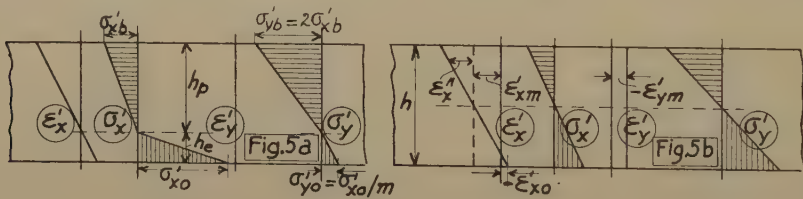


Fig. 5a and 5b.

to the right of R , and then at that side the deformation would all the same have to take place plastically. Hence the deformation is altogether plastic. If with a uniaxial compression $\varphi_1 = \sigma_v$ there is buckling perpendicular to φ_1 , ϵ'_y being 0, then, according to (10) and likewise (25) (since $\beta = 0$), in the plastic domain σ'_y would be $2\sigma'_x$, and in the elastic domain, according to equation (27) σ'_y would be σ'_x/m . As is shown by figure 5a, if $\int \sigma'_x dh$ were 0, $\int \sigma'_y dh$ would not be 0, but would create a compressive force in the Y direction. This would cause a lengthening in the Y direction, naturally accompanied by a shortening in the X direction, until the section completely reaches the plastic domain (fig. 5b). Hence at the bottom of the plate, according to equations (25) and (27) $tg \gamma_e \geq tg \gamma$, hence as $\beta = 0$ here, ϵ'_{x0} will be $\geq \frac{m-2}{2m-1} \epsilon'_y$.

the centre plane of the plate $\sigma'_x = \sigma'_y$ is to be zero. If the strains here are ϵ'_{xm} and ϵ'_{ym} , then, according to (21) and (25), $\epsilon'_{ym}/\epsilon'_{xm} = -A/B = -(1-2\beta)/(2-\beta)$. If the strains resulting from the actual bending are given by ϵ''_x and ϵ''_y , so that e.g. $\epsilon'_x = \epsilon'_{xm} + \epsilon''_x$, then it follows from (21) that:

$$\sigma'_x = E(A\epsilon''_x + B\epsilon''_y) \text{ and } \sigma'_y = E(C\epsilon''_x + D\epsilon''_y) \quad . \quad . \quad (28)$$

ϵ'_{xm} and ϵ'_{ym} will have to be the same for the entire plate, in order to make $\int \tau'_{xy} dh = \int \tau'_{yx} dh$ to be zero, for then $\gamma'_{xy} = 0$ in the centre plane.

The theory of elasticity has taught us that when w represents the deflection of the plate, ϵ''_x , ϵ''_y and γ'_{xy} , as far as the top and the bottom of the plate are concerned, are given by:

$$\epsilon''_x = -\frac{h}{2} \frac{\partial^2 w}{\partial x^2}, \quad \epsilon''_y = -\frac{h}{2} \frac{\partial^2 w}{\partial y^2} \text{ and } \gamma'_{xy} = 2 \frac{h}{2} \frac{\partial^2 w}{\partial x \partial y}.$$

Hence, according to equations (28) and (24):

$$\left. \begin{aligned} \sigma'_x &= -E \frac{h}{2} \left(A \frac{\partial^2 w}{\partial x^2} + B \frac{\partial^2 w}{\partial y^2} \right), \quad \sigma'_y = -E \frac{h}{2} \left(C \frac{\partial^2 w}{\partial x^2} + D \frac{\partial^2 w}{\partial y^2} \right) \\ \text{and } \tau'_{xy} &= \tau'_{yx} = EFh \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \right\} \quad (29)$$

The bending moments M'_x and M'_y and the twisting moments $t'_{xy} = t'_{yx}$ per unit of width, result from this by multiplication by $W = h^3/6$, as may

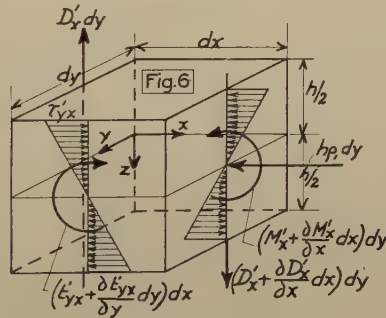


Fig. 6.

be deducted directly from either figure 5b or 6. As $h^3/12$ represents the moment of inertia I of the plate, we have:

$$\left. \begin{aligned} M'_x &= -EI \left(A \frac{\partial^2 w}{\partial x^2} + B \frac{\partial^2 w}{\partial y^2} \right), \quad M'_y = -EI \left(C \frac{\partial^2 w}{\partial x^2} + D \frac{\partial^2 w}{\partial y^2} \right) \\ \text{and } t'_{xy} &= t'_{yx} = 2EIF \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \right\} \quad (30)$$

The equilibrium of a particle $h dx dy$ requires (fig. 6):

$$D'_x = \frac{\partial M'_x}{\partial x} - \frac{\partial t'_{yx}}{\partial y} = -EI \left(A \frac{\partial^3 w}{\partial x^3} + B \frac{\partial^3 w}{\partial x \partial y^2} \right) - 2EIF \frac{\partial^3 w}{\partial x^2 \partial y^2} \quad (31a)$$

$$D'_y = \frac{\partial M'_y}{\partial y} - \frac{\partial t'_{xy}}{\partial x} = -EI \left(C \frac{\partial^3 w}{\partial x^2 \partial y} + D \frac{\partial^3 w}{\partial y^3} \right) - 2EIF \frac{\partial^3 w}{\partial x^2 \partial y^2} \quad (31b)$$

The X planes which bound the particle will rotate with respect to one another over an angle $-(\partial^2 w / \partial x^2) dx$, causing the compression forces $h \varrho_1 dy$ which act upon them, to produce a resultant force $-h \varrho_1 (\partial^2 w / \partial x^2) dx dy$ in downward direction. The same applies to ϱ_2 . The sinking of the small element is prevented by the difference between the vertical shearing forces acting upon it and by the counterpressure $c w dx dy$ of the substratum. Hence there would be equilibrium, if:

$$-\left(\frac{\partial D'_x}{\partial x} + \frac{\partial D'_y}{\partial y} \right) dx dy + c w dx dy = \left(-h \varrho_1 \frac{\partial^2 w}{\partial x^2} - h \varrho_2 \frac{\partial^2 w}{\partial y^2} \right) dx dy \quad (32)$$

or, according to (31a) and (31b):

$$EI \left\{ A \frac{\partial^4 w}{\partial x^4} + (B + C + 4F) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D \frac{\partial^4 w}{\partial y^4} \right\} + \left\{ \begin{aligned} &+ h \varrho_1 \frac{\partial^2 w}{\partial x^2} + h \varrho_2 \frac{\partial^2 w}{\partial y^2} + c w = 0 \end{aligned} \right\} \quad (33)$$

We assume ¹⁷⁾:

$$w = w_0 \cos(\pi x/a) \cos(\pi y/b) \quad (34)$$

then it follows, as $C=B$, if $\omega b=a$, and $\varrho_2=\beta \varrho_1$, that:

$$h \varrho_1 (1 + \beta \omega^2) = (\pi^2/a^2) EI \{ A + 2(B + 2F) \omega^2 + D \omega^4 \} + a^2 c / \pi^2. \quad (35)$$

Partial differentiation furnishes the buckling length a , for which ϱ_1 is minimum, viz.:

$$a = \pi [EI/c \{ A + 2(B + 2F) \omega^2 + D \omega^4 \}]^{1/4} \quad (36)$$

Then:

$$h \varrho_1 = \frac{2}{1 + \beta \omega^2} [EIc \{ A + 2(B + 2F) \omega^2 + D \omega^4 \}]^{1/2} \quad (37)$$

Differentiation shows that ϱ_1 becomes a minimum, if $\omega=0$, hence $b=\infty$. Further $d(h \varrho_1)/d\omega$ also becomes 0, if $\omega^2=(B + 2F - A\beta)/(B\beta + + 2F\beta - D)$, this value always appearing to be negative, which causes ω to become imaginary. Only if $\beta=1$, in which case there is the greatest chance of buckling in waves which cross each other, as $B=-A$

¹⁷⁾ See as well "De Ingenieur in Ned.-Indië", no. 11 and 12 (1935).

and $D=F=A$ according to (22), (23) and (24), this condition will yield 0/0, and only then buckling in waves which cross each other perpendicularly will be possible. If in that case, for example, $\omega=0$, then according to (36) $a=\pi(EIA/c)^{1/4}$ and $b=\infty$. If $\omega=\infty$, then $a=\infty$ and $b=a/\omega=\pi(EIA/c)^{1/4}$. The buckling force $h\varrho_1$ for $\beta=1$ is, according to (37), independent of ω . The expression of the coefficients B , D and F in A shows $h\varrho_1=2\sqrt{EIAc}$.

Except in the case of $\beta=1$, buckling will consequently always occur in single waves and perpendicularly to the direction of the greatest principal stress ϱ_1 , the same as in the elastic domain¹⁷⁾. Inserting $\omega=0$ in (36) and (37) gives for the (half) wave length and the buckling force:

$$a=\pi(EIA/c)^{1/4} \text{ and } h\varrho_1=2\sqrt{EIAc} \quad . \quad . \quad . \quad . \quad (38)$$

There might still be the possibility of buckling occurring in waves, the normal of which forms an angle α with ϱ_1 , according to:

$$w=w_0 \cos \frac{\pi}{a} (x+y \operatorname{tg} \alpha). \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Putting this in (33) shows:

$$h\varrho_1(1+\beta \operatorname{tg}^2 \alpha)=(\pi^2/a^2)EI\{A+2(B+2F)\operatorname{tg}^2 \alpha+D\operatorname{tg}^4 \alpha\}+a^2c/\pi^2. \quad (40)$$

This equation is identical with (35), when in the latter ω is replaced by $\operatorname{tg} \alpha$. The direct result is that $h\varrho_1$ becomes minimum if $\operatorname{tg} \alpha=0$, i.e. for waves which are perpendicular to the direction of ϱ_1 . Hence $\operatorname{tg} \alpha$ can only have an arbitrary value for $\beta=1$, and this must be so, because in that case all directions are equivalent.

Consequently, although the stress in the crust has reached the yield-value, the maximum principal stress ϱ_1 required for buckling will still have the value shown by (38). Inserting $h=3.10^6$ cm, $E=840000$ kg/cm², $I=h^3/12$ and $c=0.0023$ kg/cm² we find:

$$\varrho_1=44000\sqrt{A} \text{ kg/cm}^2 \quad . \quad . \quad . \quad . \quad . \quad (41)$$

According to (22) and (23), for β being equal to 1, $\frac{1}{2}$, 0, -1 and $-\infty$, A becomes $\frac{m}{2m+2+3em}$, 0, $\frac{m}{5m-4+3em}$, $\frac{m}{2m-2+3em}$ and $\frac{4m}{5m-4+3em}$ respectively.

This shows very clearly that the resistance to buckling, i.e. the value of A , increases with the absolute value of $\partial\sigma_x/\partial\sigma_y=\cotg \gamma$, which, according to (17), is equal to $(1-2\beta)/(2-\beta)$. For $\beta=0$, meaning a pure linear stress, with $m=5$, and with a $\sigma-\varepsilon$ diagram as in figure 3a, hence with $e=0$, we find that $A=1/4.2$. With the very improbable assumption, that even before the yield-value is reached, plastic deformations might have occurred of the tenfold magnitude of the elastic deformations¹⁸⁾, A

¹⁸⁾ The assumption in footnote 11 that $E=200000$, respectively 100000 kg/cm², would correspond here to the assumption that $e=4.8$ respectively 10.9.

becomes $1/34,2$, hence q_1 , according to (41) is reduced to 7500 kg/cm^2 , which still is considerably higher than the yield stress. By means of figure 2 it has already been stated that even in the absence of resistance to buckling, A being 0, the chance of the crust buckling before the occurrence of local plastic deformations, was slight. We may now draw the conclusion that this is impossible. A increases still for $\beta < 0$. Only if $\beta = \frac{1}{2}$, $A = 0$. As for $\beta > \frac{1}{2}$ the stress required for local plastic deformation is higher than σ_v , viz. $1,154 \sigma_v$, which makes any increase of stress impossible at buckling, there is for this reason in the domain, where $\beta \geq \frac{1}{2}$, some chance of buckling, the waves then forming themselves perpendicularly to q_1 . The strips of plastic deformation also stand here perpendicularly to q_1 , however. Consequently the deformation strips will nevertheless always occur in the directions indicated in (3) and (3a). With the further deformation of the strips, however, the increase of e , according to (40), (22), (23) and (24), will cause the stiffness to grow smaller and smaller. Owing to the increasing weight resulting from thickening the strips will moreover be apt to bend. The excentricity of the compressional force created in this way then causes them to buckle excentrically and form geosynclines¹⁹).

Bandoeng-Delft.

¹⁹) According to HOHENEMSER and PRAGER (footnote 3), with soft steel preceding "free" deformations have no influence on the state of stress. As this is not quite sure, certainly not with the crust of the earth, we took also in account the "free" deformations. If we don't do so, we only have to put $e = 0$ in our equations, which makes the crust still stronger against buckling. The excentric buckling after the formation of the strips will in this case have to be attributed to the smaller E we have to introduce in connection with "creep" (footnote 11 and 18).

Mathematics. — *Einige Integralformeln für WHITTAKERSche Funktionen.*

Von A. ERDÉLYI. (Communicated by Prof. J. G. VAN DER CORPUT.)

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§ 1. Die von Herrn C. S. MEIJER¹⁾ bewiesene Integraldarstellung eines Produktes WHITTAKERScher Funktionen

$$W_{k,m}(z) M_{-k,m}(z) = \frac{z \Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \int_0^{\infty} J_{2m}(z \sinh t) \coth^{2k} \frac{1}{2} t dt, \quad (1)$$

welche für

$$z > 0, \quad 2m \neq -1, -2, -3, \dots, \quad \Re(m-k) > -\frac{1}{2}. \quad (2)$$

gültig ist, kann vom Standpunkt der sogenannten HANKEL-Transformation gedeutet werden und führt dann zu einigen bemerkenswerten Integralformeln für das auf der linken Seite von (1) auftretende Produkt WHITTAKERScher Funktionen.

Um (1) in Gestalt einer HANKELSchen Transformierten schreiben zu können, setzen wir $z = 2\sqrt{x}$, $\sinh t = \sqrt{y}$, wodurch (1) übergeht in

$$x^{-\frac{1}{2}} W_{k,m}(2\sqrt{x}) M_{-k,m}(2\sqrt{x}) = \left. \begin{aligned} &= \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \int_0^{\infty} J_{2m}(2\sqrt{xy}) \frac{(1+\sqrt{1+y})^{2k}}{y^{k+\frac{1}{2}}\sqrt{1+y}} dy \end{aligned} \right\} \quad (3)$$

§ 2. Bekanntlich nennt man

$$F(x) = \int_0^{\infty} J_{\nu}(2\sqrt{xy}) f(y) dy \quad (4)$$

die HANKELSche Transformierte ν -ter Ordnung von $f(x)$. Unter gewissen Voraussetzungen gilt die Umkehrformel

$$f(x) = \int_0^{\infty} J_{\nu}(2\sqrt{xy}) F(y) dy, \quad (5)$$

¹⁾ C. S. MEIJER, Noch einige Integraldarstellungen für Produkte von WHITTAKERSchen Funktionen, Gleichung (4). Proc. Royal Acad. Amsterdam, **40**, 871—879 (1937).

und ferner besteht die folgende Beziehung zwischen den LAPLACE-Transformierten von $x^{\frac{1}{2}\nu} F(x)$ und $x^{\frac{1}{2}\nu} f(x)$ ²⁾:

$$s^{\frac{1}{2}(\nu+1)} \int_0^{\infty} e^{-sx} x^{\frac{1}{2}\nu} f(x) dx = \left(\frac{1}{s}\right)^{\frac{1}{2}(\nu+1)} \int_0^{\infty} e^{-\frac{1}{s}x} x^{\frac{1}{2}\nu} F(x) dx \quad [\Re(s) > 0] \quad (6)$$

§ 3. Für das Funktionenpaar

$$\left. \begin{aligned} f(x) &= \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \frac{(1+\sqrt{1+x})^{2k}}{x^{k+\frac{1}{2}}\sqrt{1+x}} \\ F(x) &= x^{-\frac{1}{2}} W_{k,m}(2\sqrt{x}) M_{-k,m}(2\sqrt{x}) \end{aligned} \right\} \quad \dots \quad (7)$$

sind die in § 2 erwähnten Voraussetzungen (mit $\nu=2m$) erfüllt, wenn

$$\Re(m) > -\frac{1}{2}, \quad \Re(k) < \frac{1}{4}, \quad \Re(m-k) > -\frac{1}{2} \quad \dots \quad (8)$$

und x positiv reell ist ($\sqrt{x}$ bedeutet die positiv ausgezogene Quadratwurzel). Diese Annahmen sollen von nun an beibehalten werden. Sind sie erfüllt, dann ist auch (2) erfüllt, und daher gilt (3). Die Umkehrung von (3) nach (5) ergibt die Integralformel

$$\boxed{\begin{aligned} \int_0^{\infty} J_{2m}(2\sqrt{xy}) W_{k,m}(2\sqrt{y}) M_{-k,m}(2\sqrt{y}) \frac{dy}{\sqrt{y}} &= \\ &= \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \frac{(1+\sqrt{1+x})^{2k}}{x^{k+\frac{1}{2}}\sqrt{1+x}} \quad [\Re(m) > -\frac{1}{2}, \Re(k) < \frac{1}{4}]. \end{aligned}} \quad \dots \quad (9)$$

Gewissermassen ein Gegenstück zu (9), die Integralformel

$$\int_0^{\infty} J_{4m}(2\sqrt{xy}) W_{k,m}(y) M_{-k,m}(y) \frac{dy}{y} = \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} K_{m+k}(\frac{1}{2}x) I_{m-k}(\frac{1}{2}x) \left\{ \begin{aligned} &[\Re(m) > -\frac{1}{4}, \Re(k) < \frac{1}{8}] \end{aligned} \right\} \quad (10)$$

werde ich an anderer Stelle beweisen.

Die Anwendung der Identität (6) auf das Funktionenpaar (7) liefert die Beziehung

$$\boxed{\begin{aligned} \int_0^{\infty} e^{-sx} x^{m-\frac{1}{2}} W_{k,m}(2\sqrt{x}) M_{-k,m}(2\sqrt{x}) dx &= \\ &= \frac{2\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} s^{-2m-1} \int_0^{\infty} e^{-\frac{1}{s}\xi(\xi+2)} \xi^{-\frac{1}{2}-k+m} (\xi+2)^{-\frac{1}{2}+k+m} d\xi \\ &[\Re(m) > -\frac{1}{2}, \Re(m-k) > -\frac{1}{2}, \Re(s) > 0], \end{aligned}} \quad \dots \quad (11)$$

²⁾ Vergl. z.B.: F. TRICOMI, Sulla trasformazione e il teorema di reciprocità di HANKEL. Rendiconti dei Lincei 22, 564—571 (1935).

wobei auf der rechten Seite die neue Integrationsveränderliche $\xi = \sqrt{1+x} - 1$ eingeführt wurde.

§ 4. Einige Sonderfälle der Gleichungen (9) und (11) verdienen besonders angeschrieben zu werden. Das gilt vor allem von jenen Fällen, in denen die auf den linken Seiten von (9) und (11) auftretenden WHITTAKERSchen Funktionen sich durch BESSELSche ausdrücken lassen. Bekanntlich ist ³⁾

$$\left. \begin{aligned} W_{0,m}(2\sqrt{z}) &= \sqrt{\frac{2}{\pi}} z^{\frac{1}{2}} K_m(\sqrt{z}) \\ M_{0,m}(2\sqrt{z}) &= 2^{2m+\frac{1}{2}} z^{\frac{1}{2}} \Gamma(m+1) I_m(\sqrt{z}). \end{aligned} \right\} \dots (12)$$

Daher ergibt die Annahme $k=0$ in (9) nach Anwendung der Verdopplungsformel der Gammafunktion

$$\Gamma(2m+1) = \frac{1}{\sqrt{\pi}} 2^{2m} \Gamma(m+\frac{1}{2}) \Gamma(m+1)$$

die für $\Re(m) > -\frac{1}{2}$ geltende Beziehung ⁴⁾

$$\int_0^\infty J_{2m}(2\sqrt{xy}) K_m(\sqrt{y}) I_m(\sqrt{y}) dy = \frac{1}{2\sqrt{x(1+x)}}, \quad (13)$$

während dieselbe Annahme in (11) zunächst die Identität

$$\int_0^\infty e^{-sx} x^m K_m(\sqrt{x}) I_m(\sqrt{x}) dx = \frac{1}{s} s^{-2m-1} \int_0^\infty e^{-\frac{1}{s}x} x^{m-\frac{1}{2}} \frac{dx}{\sqrt{1+x}}$$

liefert. Das auf der rechten Seite stehende Integral kann mit Hilfe einer bekannten Integraldarstellung der WHITTAKERSchen Funktion, die in unserem besonderen Falle

$$\int_0^\infty e^{-\frac{1}{s}x} x^{m-\frac{1}{2}} (1+x)^{-\frac{1}{2}} dx = \Gamma(m+\frac{1}{2}) s^{\frac{1}{2}m+\frac{1}{2}} e^{\frac{1}{2s}} W_{-\frac{1}{2}m, \frac{1}{2}m}\left(\frac{1}{s}\right)$$

lautet ⁵⁾, leicht ausgerechnet werden und ergibt

$$\int_0^\infty e^{-sx} x^m K_m(\sqrt{x}) I_m(\sqrt{x}) dx = \frac{1}{s} \Gamma(m+\frac{1}{2}) s^{-\frac{3}{2}m-\frac{1}{2}} e^{\frac{1}{2s}} W_{-\frac{1}{2}m, \frac{1}{2}m}\left(\frac{1}{s}\right) \left\{ \begin{aligned} (14) \\ [\Re(m) > -\frac{1}{2}, \Re(s) > 0]. \end{aligned} \right.$$

³⁾ Vergl. z.B.: MEIJER, l.c., § 7.

⁴⁾ Vergl. G. N. WATSON, Theory of BESSEL functions, § 13. 6, Gleichung (3). Cambridge (1922).

⁵⁾ E. T. WHITTAKER and G. N. WATSON, A course of modern analysis, 4th Ed., § 16. 12. Cambridge (1927).

§ 5. Weitere erwähnenswerte Sonderfälle von (9) und (11) sind jene, in denen sich die WHITTAKERSchen Funktionen auf die *Funktionen des parabolischen Zylinders* reduzieren. Bekanntlich ist ⁶⁾

$$W_{\frac{1}{2}\nu + \frac{1}{4}, \pm \frac{1}{4}}(2\sqrt{z}) = 2^{-\frac{1}{2}\nu + \frac{1}{4}} z^{1/8} D_{\nu}(2z^{\frac{1}{4}}),$$

und daher

$$M_{-\frac{1}{2}\nu - \frac{1}{4}, \frac{1}{4}}(2\sqrt{z}) = \frac{2^{\frac{1}{2}\nu - \frac{1}{4}} \Gamma(\frac{1}{2}\nu + \frac{1}{2})}{\Gamma(-\frac{1}{2})} z^{1/8} [D_{-\nu-1}(2z^{\frac{1}{4}}) - D_{-\nu-1}(-2z^{\frac{1}{4}})],$$

$$M_{-\frac{1}{2}\nu - \frac{1}{4}, -\frac{1}{4}}(2\sqrt{z}) = \frac{2^{\frac{1}{2}\nu - \frac{1}{4}} \Gamma(\frac{1}{2}\nu + 1)}{\Gamma(\frac{1}{2})} z^{1/8} [D_{-\nu-1}(2z^{\frac{1}{4}}) + D_{-\nu-1}(-2z^{\frac{1}{4}})].$$

Da ferner

$$J_{\frac{1}{2}}(2\sqrt{xy}) = \frac{1}{\sqrt{\pi}} (xy)^{-\frac{1}{4}} \sin 2\sqrt{xy}$$

und

$$J_{-\frac{1}{2}}(2\sqrt{xy}) = \frac{1}{\sqrt{\pi}} (xy)^{-\frac{1}{4}} \cos 2\sqrt{xy}$$

ist, so ergibt die Annahme $m = \frac{1}{4}$, $k = \frac{1}{2}\nu + \frac{1}{4}$ in (9)

$$\left. \begin{aligned} \int_0^{\infty} \sin 2\sqrt{xy} \cdot D_{\nu}(2y^{\frac{1}{4}}) [D_{-\nu-1}(2y^{\frac{1}{4}}) - D_{-\nu-1}(-2y^{\frac{1}{4}})] \frac{dy}{\sqrt{y}} = \\ = -\sqrt{\pi} \cos \frac{1}{2}\nu\pi \frac{(1 + \sqrt{1+x})^{\nu+\frac{1}{2}}}{x^{\frac{1}{2}\nu+\frac{1}{4}} \sqrt{1+x}}, \end{aligned} \right\} \quad (15)$$

$[\Re(\nu) < 0],$

während die Annahme $m = -\frac{1}{4}$, $k = \frac{1}{2}\nu + \frac{1}{4}$ zu der verwandten Beziehung

$$\left. \begin{aligned} \int_0^{\infty} \cos 2\sqrt{xy} \cdot D_{\nu}(2y^{\frac{1}{4}}) [D_{-\nu-1}(2y^{\frac{1}{4}}) + D_{-\nu-1}(-2y^{\frac{1}{4}})] \frac{dy}{\sqrt{y}} = \\ = -\sqrt{\pi} \sin \frac{1}{2}\nu\pi \frac{(1 + \sqrt{1+x})^{\nu+\frac{1}{2}}}{x^{\frac{1}{2}\nu+\frac{1}{4}} \sqrt{1+x}} \end{aligned} \right\} \quad (16)$$

$[\Re(\nu) < 0]$

führt. Mit denselben Annahmen ergeben sich aus (11) folgende beiden Identitäten:

⁶⁾ Modern Analysis, § 16.5.

$$\left. \begin{aligned} & \int_0^{\infty} e^{-sx} D_{\nu}(2x^{\frac{1}{2}}) [D_{-\nu-1}(2x^{\frac{1}{2}}) - D_{-\nu-1}(-2x^{\frac{1}{2}})] dx = \\ & = -2s^{-3/2} \cos \frac{1}{2} \nu \pi \int_0^{\infty} e^{-\frac{1}{s} \xi (\xi + 2)} \left(1 + \frac{2}{\xi}\right)^{\frac{1}{2} \nu} \frac{d\xi}{\sqrt{\xi}} = \\ & [\Re(\nu) < 1, \Re(s) > 0], \end{aligned} \right\} \quad (17)$$

$$\left. \begin{aligned} & \int_0^{\infty} e^{-sx} D_{\nu}(2x^{\frac{1}{2}}) [D_{-\nu-1}(2y^{\frac{1}{2}}) + D_{-\nu-1}(-2y^{\frac{1}{2}})] \frac{dx}{\sqrt{x}} = \\ & = -2s^{-\frac{1}{2}} \sin \frac{1}{2} \nu \pi \int_0^{\infty} e^{-\frac{1}{s} \xi (\xi + 2)} \left(1 + \frac{2}{\xi}\right)^{\frac{1}{2} \nu - \frac{1}{2}} \frac{d\xi}{\xi^{3/2}} = \\ & [\Re(\nu) < 0, \Re(s) > 0]. \end{aligned} \right\} \quad (18)$$

Bei der Ableitung von (15)–(18) wurde zur Vereinfachung der „Ergänzungssatz“ der Theorie der Gammafunktion

$$\Gamma\left(\frac{1}{2} + w\right) \Gamma\left(\frac{1}{2} - w\right) = \frac{\pi}{\cos w \pi} = \frac{\pi}{\sin\left(\frac{1}{2} \pm w\right) \pi}$$

verwendet. Diese Identitäten sind auch deswegen interessant, weil sie eine Beziehung zwischen den linken Seiten von (15) und (16) aufzeigen, deren besondere Aufschreibung wohl überflüssig ist.

§ 6. Als letztes Beispiel für durch Besonderung aus (9) und (11) hervorgehende Beziehungen sollen einige LAGUERRESche Polynome enthaltenden Formeln angeführt werden. Der Zusammenhang zwischen diesen Polynomen und den WHITTAKERSchen Funktionen wird z.B. durch die Formel⁷⁾

$$M_{m+n+\frac{1}{2},m}(2\sqrt{z}) = \frac{\Gamma(1+2m)n!}{\Gamma(1+2m+n)} (2\sqrt{z})^{m+\frac{1}{2}} e^{-\sqrt{z}} L_n^{(2m)}(2\sqrt{z})$$

gegeben. Wenn wir in (9) und (11) $k = -\frac{1}{2} - m - n$ ($n = 0, 1, 2, \dots$) setzen, so folgen unmittelbar die beiden Identitäten

$$\left. \begin{aligned} & \int_0^{\infty} J_{2m}(2\sqrt{xy}) W_{-\frac{1}{2}-m-n,m}(2\sqrt{y}) L_n^{(2m)}(2\sqrt{y}) (2\sqrt{y})^{m-\frac{1}{2}} e^{-\sqrt{y}} dy \\ & = \frac{1}{2 \cdot n!} \frac{x^{m+n} (1+x)^{-\frac{1}{2}}}{(1+\sqrt{1+x})^{1+2m+2n}} [\Re(m) > -\frac{1}{2}, n=0, 1, 2, \dots], \end{aligned} \right\} \quad (19)$$

⁷⁾ Vergl. z.B.: A. ERDÉLYI, Funktionalrelationen mit konfluenten hypergeometrischen Funktionen. Erste Mitteilung: Additions- und Multiplikationstheoreme, Gleichungen (1, 9) und (1, 4). Math. Zeitschr. **42**, 125–143 (1936).

$$\left. \begin{aligned} & \int_0^{\infty} e^{-sx} \sqrt{x} x^{\frac{3}{2}m-\frac{1}{2}} W_{-\frac{1}{2}-m-n, m}(2\sqrt{x}) L_n^{(2m)}(2\sqrt{x}) dx \\ &= \frac{2^{\frac{1}{2}-m}}{n!} s^{-2m-1} \int_0^{\infty} e^{-\frac{1}{s}\xi(\xi+2)} \xi^{2m+n} (\xi+2)^{-1-n} d\xi \\ & [\Re(m) > -\frac{1}{2}, \Re(s) > 0, n=0, 1, 2, \dots] \end{aligned} \right\} \quad (20)$$

§ 7. Viele der von Herrn MEIJER in den letzten Jahren publizierten zahlreichen Integraldarstellungen — und auch ähnlich gebaute ältere Integraldarstellungen — können auf ganz ähnliche Weise behandelt werden und ergeben Integralformeln, in denen ausser den WHITTAKERSchen Funktionen und ihren Sonderfällen, wie BESSELSchen Funktionen, Funktionen des parabolischen Zylinders usw., noch LOMMELSche und STRUVESche Funktionen, hypergeometrische Reihen, Kugelfunktionen usw. auftreten. Da die Herleitung aller auf diese Weise zu gewinnender Integralformeln in gleicher Weise verläuft wie die Herleitung von (9) und (11), so genügt es, an dieser Stelle auf diese Möglichkeit bloss hinzuweisen, ohne die betreffenden Formeln ausführlich zu notieren. Die Anzahl der für diese Behandlung geeigneten Integraldarstellungen ist so gross, dass sie hier nicht einmal aufgezählt werden können. Es möge aber wenigstens darauf hingewiesen werden, dass insbesondere auch die sehr allgemeine Integraldarstellung ⁸⁾

$$\begin{aligned} & B_{p,r} \left(x \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_{p+2} \end{array} \right. \right) \\ &= x^{\frac{1}{2}(\beta-\alpha+1)} \int_0^{\infty} y^{\frac{1}{2}(\beta-\alpha-1)} J_{\alpha+\beta-1}(2\sqrt{xy}) \Psi_{p,p-r+1} \left(y \left| \begin{array}{c} 1-\alpha, a_1, \dots, a_p, \beta \\ b_1, \dots, b_{p+2} \end{array} \right. \right) dy \end{aligned}$$

mit ihren zahlreichen wichtigen Sonderfällen zu dieser Klasse von Integralen gehört.

⁸⁾ C. S. MEIJER, Einige Integraldarstellungen aus der Theorie der BESSELSchen und WHITTAKERSchen Funktionen. Proc. Royal Acad. Amsterdam, 39, 394—403 und 519—527 (1936); Gleichung (10) in etwas anderer Schreibweise. Wegen der Bezeichnungen und der Gültigkeitsbedingungen dieser Integraldarstellung muss hier auf die angeführte Arbeit selbst verwiesen werden.

Mathematics. — *On the iteration of linear operators in a HILBERT space.* By CORNELIS VISSER. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of April 30, 1938).

Let $\mathfrak{H}$ be a HILBERT space. Its elements will be denoted by $x, y, \dots$, the inner product of x and y by (x, y) , the norm of x by $|x|$. It is not assumed that $\mathfrak{H}$ satisfies the separability postulate.

In some applications of the theory of linear operators the problem arises whether for a given linear operator A in $\mathfrak{H}$ and an arbitrary element x the arithmetic mean

$$\frac{1}{n} (Ax + A^2x + \dots + A^n x)$$

has a limit when n approaches infinity. In the present paper it will be proved that there exists always a limit in the sense of weak convergence, it being assumed that the operators A^n ($n = 1, 2, \dots$) are equibounded. In some important special cases it is easy to establish *a posteriori* that the convergence is strong. Thus we shall find as corollaries a theorem of J. VON NEUMANN¹⁾ on unitary operators which is the basis of his "quasi-ergodic" theorem and a generalization of a theorem due to M. FRÉCHET²⁾ on the iteration of a FREDHOLM kernel.

1. *Weak and strong convergence.*

We shall have to deal with two kinds of convergence in the HILBERT space $\mathfrak{H}$, weak convergence and strong convergence. I briefly recall some relevant facts.

A sequence of elements x_n ($n = 1, 2, \dots$) is said to be *strongly convergent* towards the element z if $|x_n - z| \rightarrow 0$ when $n \rightarrow \infty$. The sequence is said to be *weakly convergent* towards z if $(x_n, y) \rightarrow (z, y)$ when $n \rightarrow \infty$ for any element y . Strong convergence implies weak convergence; the converse is not true.

If a sequence x_n ($n = 1, 2, \dots$) is bounded, then it contains a weakly convergent subsequence.

If $x_n \rightarrow z$ weakly, and if A is a bounded linear operator, then $Ax_n \rightarrow Az$ weakly. For, if A^* denotes the adjoint of A , we have

¹⁾ J. VON NEUMANN, Proof of the quasi-ergodic hypothesis, Proc. Nat. Acad. U.S.A., 18, 70—82 (1932).

²⁾ M. FRÉCHET, Sur l'allure asymptotique de la suite des itérés d'un noyau de Fredholm, Quart. J. of Math., Oxford Ser., 5, 106—144 (1934).

$x_n, A^*y) \rightarrow (z, A^*y)$ for any element y , and this is equivalent to $(Ax_n, y) \rightarrow (Az, y)$ for any element y .

If $x_n \rightarrow z$ weakly, and if all x_n belong to a compact set, then $x_n \rightarrow z$ strongly. For otherwise there would exist a positive number ε and an infinite subsequence $x'_n (n=1, 2, \dots)$ of the given sequence x_n such that $|x'_n - z| \geq \varepsilon$ for all $n=1, 2, \dots$. Since the x_n belong to a compact set the sequence x'_n would have an infinite subsequence $x''_n (n=1, 2, \dots)$ converging strongly towards a certain element u . We should have $|u - z| \geq \varepsilon$; on the other hand $(x''_n - z, u - z) \rightarrow 0$ and $x''_n \rightarrow u$ strongly imply $(u - z, u - z) = 0$, which is a contradiction.

Finally I observe that a bounded sequence $x_n (n=1, 2, \dots)$ which does not converge weakly towards 0 has a subsequence converging weakly towards an element different from 0. For in that case there exist a positive number ε , an element y , and a subsequence $x'_n (n=1, 2, \dots)$ such that $|(x'_n, y)| \geq \varepsilon$ for all n . The sequence x'_n contains a subsequence x''_n converging weakly towards a certain element z . We have $(x''_n, y) \rightarrow (z, y)$ when $n \rightarrow \infty$ and hence $|(z, y)| \geq \varepsilon$, which shows that $z \neq 0$.

2. Operators for which 1 is not a proper value.

We shall first take into consideration a bounded linear operator A for which 1 is not a proper value, that is to say the equation $Ax = x$ is only satisfied by $x = 0$. We assume that there exists a real number M such that

$$|A^n x| \leq M |x|$$

for any element x and $n=1, 2, \dots$. We shall express this fact by saying that the operators $A^n (n=1, 2, \dots)$ are *equi-bounded*.

Taking a fixed element x we put

$$y_n = \frac{1}{n} (Ax + A^2x + \dots + A^n x) \quad (n=1, 2, \dots).$$

Now suppose that the sequence y_n did not tend weakly towards 0. Clearly the sequence y_n is bounded. Hence there would exist a subsequence $y_{n_i} (i=1, 2, \dots)$ converging weakly towards an element z different from 0. Our hypothesis implies that

$$|Ay_n - y_n| \rightarrow 0$$

when $n \rightarrow \infty$. It follows that also

$$Ay_{n_i} \rightarrow z \text{ weakly}$$

when $i \rightarrow \infty$. On the other hand, $y_{n_i} \rightarrow z$ weakly implies $Ay_{n_i} \rightarrow Az$ weakly. Hence $Az = z$, which contradicts the assumption that 1 is not a proper value. Thus we have proved that for any element x

$$\frac{1}{n} (Ax + A^2x + \dots + A^n x) \rightarrow 0 \text{ weakly}$$

when $n \rightarrow \infty$.

3. *A lemma.*

We shall next prove a lemma. It follows immediately from what we just have proved. Consider the adjoint A^* of A . The adjoint of A^n is A^{*n} , that of $\frac{1}{n} (A + A^2 + \dots + A^n)$ is $\frac{1}{n} (A^* + A^{*2} + \dots + A^{*n})$. Since a bounded linear operator and its adjoint have the same bound, we have also

$$|A^{*n}x| \leq M|x|$$

for all x and $n=1, 2, \dots$. Now we have proved that for any x and y

$$\left(\frac{1}{n} (Ax + A^2x + \dots + A^nx), y \right) \rightarrow 0$$

when $n \rightarrow \infty$. Another way of stating this is

$$\left(x, \frac{1}{n} (A^*y + A^{*2}y + \dots + A^{*n}y) \right) \rightarrow 0$$

when $n \rightarrow \infty$. Now suppose that $A^*y = y$. Then we find on putting $x=y$

$$(y, y) \rightarrow 0$$

and hence $y=0$. Consequently A^* does not have 1 as a proper value.

Clearly the parts played by A and A^* in our reasoning are interchangeable. We may therefore state the following

Lemma. *If the operators A^n ($n=1, 2, \dots$) are equi-bounded, then 1 is a proper value either for both A and A^* or for none of them.*

4. *Proof of the main theorem.*

We now turn to the general case. Our only assumption about the linear operator A is:

$$|A^n x| \leq M|x|$$

for any element x and $n=1, 2, \dots$.

Let $\mathfrak{M}$ denote the closed linear manifold formed by all x with $Ax=x$, $\mathfrak{M}_1$ its orthogonal complement, $\mathfrak{M}^*$ the closed linear manifold formed by all x with $A^*x=x$, $\mathfrak{M}_1^*$ its orthogonal complement. Let P, P_1, Q, Q_1 denote the projections on $\mathfrak{M}, \mathfrak{M}_1, \mathfrak{M}^*, \mathfrak{M}_1^*$ respectively.

A^* transforms $\mathfrak{M}_1$ into itself. For $(x, y)=0$ for all $x \in \mathfrak{M}$ implies $(x, A^*y)=(Ax, y)=(x, y)=0$ for all $x \in \mathfrak{M}$. If we consider A^* as an operator in the space $\mathfrak{M}_1$, its adjoint is $P_1 A$. For if $x \in \mathfrak{M}_1, y \in \mathfrak{M}_1$, we have

$$(A^*x, y) = (x, Ay) = (P_1x, Ay) = (x, P_1Ay)$$

and of course $P_1Ay \in \mathfrak{M}_1$. Now it is easily seen that P_1A has not 1 as a proper value. For otherwise for some $x \neq 0$ $P_1Ax=x$. Then $Ax - x \in \mathfrak{M}$. Hence

$$Ax - x = A^2x - Ax = A^3x - A^2x = \dots$$

Owing to the equi-boundedness of the operators A^n this implies $Ax=x$ which contradicts the fact that, because of $P_1 Ax=x$, x must belong to $\mathfrak{M}_1$. It follows, by the preceding lemma, that A^* considered as an operator in $\mathfrak{M}_1$ has not 1 as a proper value. Consequently $\mathfrak{M}_1$ intersects $\mathfrak{M}^*$ only in the zero-element. Interchanging the parts of A and A^* we find that also $\mathfrak{M}_1^*$ and $\mathfrak{M}$ have only the zero-element in common.

We next show that Q maps $\mathfrak{M}$ in a one-to-one manner on the whole of $\mathfrak{M}^*$. If the mapping were not one-to-one there would exist an element $z \in \mathfrak{M}$, $z \neq 0$, such that $Qz=0$. Hence z would belong to $\mathfrak{M}_1^*$ which contradicts the relation $\mathfrak{M}_1^* \mathfrak{M}=0$. If Q did not map $\mathfrak{M}$ on the whole of $\mathfrak{M}^*$ there would exist an element $z \in \mathfrak{M}^*$, $z \neq 0$, such that $(Qx, z)=0$ for all $x \in \mathfrak{M}$. Hence $(x, Qz)=0$ and, since $Qz=z$, $(x, z)=0$ for all $x \in \mathfrak{M}$. This means $z \in \mathfrak{M}_1$, which contradicts the relation $\mathfrak{M}_1 \mathfrak{M}^*=0$.

Let R denote the inverse of this mapping. We take an arbitrary element x and put

$$x_1 = RQx, \quad x_2 = x - RQx.$$

x_1 belongs to $\mathfrak{M}$; x_2 belongs to $\mathfrak{M}_1^*$ since

$$Qx_2 = Qx - QRQx = Qx - Qx = 0.$$

A transforms $\mathfrak{M}$ and $\mathfrak{M}_1^*$ into themselves. Hence

$$A^n x = A^n x_1 + A^n x_2 = x_1 + A^n x_2.$$

If we consider A as an operator in $\mathfrak{M}_1^*$ it has not 1 as a proper value and satisfies, of course, our fundamental assumption. Therefore

$$\frac{1}{n} (Ax_2 + A^2x_2 + \dots + A^n x_2) \rightarrow 0 \text{ weakly}$$

when $n \rightarrow \infty$ and consequently

$$\frac{1}{n} (Ax + A^2x + \dots + A^n x) \rightarrow RQx \text{ weakly}$$

when $n \rightarrow \infty$. We have thus proved the following

Theorem. *If A is a linear operator and the operators A^n ($n=1, 2, \dots$) are equi-bounded, then there exists a bounded linear operator L such that*

$$\frac{1}{n} (Ax + A^2x + \dots + A^n x) \rightarrow Lx \text{ weakly}$$

when $n \rightarrow \infty$ for any element x .

We have put here $RQ=L$. That L is a bounded linear operator is obvious. It satisfies the relation $AL=L$.

5. Isometric operators.

We shall consider in this section an isometric operator U , i. e. a

linear operator with the property that

$$(Ux, Uy) = (x, y)$$

for all x and y . It clearly satisfies our fundamental assumption. We shall only need the result of sec 2.

Suppose first that 1 is not a proper value. Then

$$\frac{1}{n} (Ux + U^2x + \dots + U^nx) \rightarrow 0 \text{ weakly}$$

when $n \rightarrow \infty$. We shall show that this convergence is strong owing to the fact that U is isometric. We put, x being fixed,

$$s_n = (Ux, Ux) + (U^2x, Ux) + \dots + (U^nx, Ux) \quad (n=1, 2, \dots).$$

Then

$$\frac{s_n}{n} \rightarrow 0$$

when $n \rightarrow \infty$. Now

$$\begin{aligned} \left| \frac{1}{n} (Ux + U^2x + \dots + U^nx) \right|^2 &= \frac{1}{n^2} \sum_{p=1}^n \sum_{q=1}^n (U^p x, U^q x) \\ &= -\frac{1}{n^2} \sum_{p=1}^n |U^p x|^2 + \frac{2}{n^2} \Re \sum_{q=1}^n \sum_{p=q}^n (U^p x, U^q x) \\ &= -\frac{|x|^2}{n} + \frac{2}{n^2} \Re \sum_{q=1}^n \sum_{p=q}^n (U^{p-q+1} x, Ux) \\ &= -\frac{|x|^2}{n} + \frac{2}{n^2} \Re \sum_{q=1}^n s_{n-q+1}, \end{aligned}$$

and this clearly tends to 0 when $n \rightarrow \infty$.

Now let 1 be a proper value. Let $\mathfrak{M}$ be the closed linear manifold formed by all elements x for which $Ux = x$, $\mathfrak{M}_1$ its orthogonal complement. Let P and P_1 denote the projections on $\mathfrak{M}$ and $\mathfrak{M}_1$ respectively. U transforms $\mathfrak{M}_1$ into itself. If we consider U as an operator in $\mathfrak{M}_1$ it is an isometric operator for which 1 is not a proper value. Now take x arbitrary and put $x_1 = Px$, $x_2 = P_1x$. Then, by what we just have proved,

$$\frac{1}{n} (Ux_2 + U^2x_2 + \dots + U^nx_2) \rightarrow 0 \text{ strongly}$$

when $n \rightarrow \infty$. Since $U^nx = U^nx_1 + U^nx_2 = x_1 + U^nx_2$, it follows that

$$\frac{1}{n} (Ux + U^2x + \dots + U^nx) \rightarrow x_1 \text{ strongly}$$

when $n \rightarrow \infty$. We have thus obtained the following result.

If U is an isometric operator, P the projection on the closed linear manifold formed by all x with $Ux = x$, then

$$\frac{1}{n}(Ux + U^2x + \dots + U^nx) \rightarrow Px \text{ strongly}$$

when $n \rightarrow \infty$ for any element x .

J. VON NEUMANN proved the following theorem¹⁾ from which his quasi-ergodic theorem is a direct consequence.

Let U_t ($-\infty < t < \infty$) be a family of unitary operators with the group property

$$(a) \quad U_t U_s = U_{t+s} \text{ for all real } t \text{ and } s,$$

and the measurability property

(b) $(U_t x, y)$ is a measurable function of t for arbitrary elements x and y .

Then for any element x

$$\frac{1}{T} \int_a^{a+T} U_t x \, dt$$

tends strongly to a limit when $T \rightarrow \infty$, uniformly for $-\infty < a < \infty$.

Here the integral is defined in the following manner. It is clear that

$\int_a^{a+T} (y, U_t x) \, dt$ is a bounded linear functional of y . Hence, by a well-known

theorem, there exists a unique element z such that $\int_a^{a+T} (y, U_t x) \, dt = (y, z)$.

Now $\int_a^{a+T} U_t x \, dt$ is defined to be this element z . This sort of integral

has the ordinary properties of the LEBESGUE integral.

The proof that VON NEUMANN gave of his theorem is based upon the spectral resolution of the operators U_t . Another, simpler, proof was devised by E. HOPF²⁾. Our preceding result enables us to give the following proof. We put, x being fixed but arbitrary,

$$y = \int_0^1 U_t x \, dt, \quad U = U_1.$$

Let P be the projection on the closed linear manifold formed by all x with $Ux = x$. Then

$$\frac{1}{n} \int_0^n U_t x \, dt = \frac{y + Uy + \dots + U^{n-1}y}{n} \rightarrow Py \text{ strongly}$$

¹⁾ Loc. cit.

²⁾ E. HOPF, On the time average theorem in dynamics, Proc. Nat. Acad. U.S.A., 18, 93-100 (1932).

when $n \rightarrow \infty$ through integer values. Now write $T = n + \tau$, n integer, $0 \leq \tau < 1$. It is plain that

$$\frac{1}{n + \tau} \int_0^{n + \tau} U_t x \, dt - \frac{1}{n} \int_0^n U_t x \, dt \rightarrow 0 \text{ strongly}$$

when $n \rightarrow \infty$ (τ arbitrary). Therefore

$$\frac{1}{T} \int_0^T U_t x \, dt \rightarrow Py \text{ strongly}$$

when $T \rightarrow \infty$. We have further

$$U^k \left(\frac{1}{T} \int_0^T U_t x \, dt - Py \right) = \frac{1}{T} \int_k^{k+T} U_t x \, dt - Py.$$

It follows that

$$\frac{1}{T} \int_k^{k+T} U_t x \, dt \rightarrow Py \text{ strongly,}$$

uniformly for $k = 0, \pm 1, \pm 2, \dots$, and from this it follows easily that

$$\frac{1}{T} \int_a^{a+T} U_t x \, dt \rightarrow Py \text{ strongly}$$

uniformly for $-\infty < a < \infty$.

6. Symmetric linear operators.

Let A be a symmetric linear operator, that is to say a linear operator which satisfies the relation

$$(Ax, y) = (x, Ay)$$

for all x and y . We suppose that the operators A^n ($n = 1, 2, \dots$) are equi-bounded. If 1 is not a proper value, then

$$\frac{1}{n} (Ax + A^2x + \dots + A^n x) \rightarrow 0 \text{ weakly.}$$

Also here we can show that the convergence is strong. We put

$$s_n = (Ax, Ax) + (A^2x, Ax) + \dots + (A^n x, Ax).$$

Then

$$\frac{s_n}{n} \rightarrow 0$$

when $n \rightarrow \infty$. Now

$$\begin{aligned} \left| \frac{1}{n} (Ax + A^2x + \dots + A^n x) \right|^2 &= \frac{1}{n^2} \sum_{p=1}^n \sum_{q=1}^n (A^p x, A^q x) \\ &= \frac{1}{n^2} \sum_{p=1}^n \sum_{q=1}^n (A^{p+q-1} x, Ax) \\ &= \frac{s_n + (s_{n+1} - s_1) + (s_{n+2} - s_2) + \dots + (s_{2n-1} - s_{n-1})}{n^2} \\ &= \frac{s_1 + s_2 + \dots + s_{2n-1}}{n^2} - \frac{2(s_1 + s_2 + \dots + s_{n-1})}{n^2} \rightarrow 0 \end{aligned}$$

when $n \rightarrow \infty$.

If 1 is a proper value, we introduce again the projection P on the closed linear manifold formed by all x with $Ax = x$. Using the well-known fact that A transforms the orthogonal complement of this manifold into itself we find as in sec 5 that

$$\frac{1}{n} (Ax + A^2x + \dots + A^n x) \rightarrow Px \text{ strongly}$$

when $n \rightarrow \infty$.

7. Totally continuous linear operators.

A linear operator is said to be totally continuous if it transforms any bounded set into a conditionally compact (totally bounded) set. Suppose that A is totally continuous and again that $|A^n x| \leq M|x|$ for all x and $n = 1, 2, \dots$. We proved the existence of a bounded linear operator L such that

$$\frac{1}{n} (Ax + A^2x + \dots + A^n x) \rightarrow Lx \text{ weakly}$$

when $n \rightarrow \infty$ for any x . The total continuity of A involves that this convergence is strong. Let $\mathfrak{A}$ denote the set of all Ax with $|x| \leq 1$. Since

$$\frac{1}{n} (Ax + A^2x + \dots + A^n x) = A \cdot \frac{1}{n} (x + Ax + \dots + A^{n-1} x)$$

and

$$\left| \frac{1}{n} (x + Ax + \dots + A^{n-1} x) \right| \leq M$$

for $|x| \leq 1$, the element $\frac{1}{n} (Ax + A^2x + \dots + A^n x)$ belongs $M\mathfrak{A}$ for any x with $|x| \leq 1$ and $n = 1, 2, \dots$. Here $M\mathfrak{A}$ is the set of all x with $\frac{1}{M}x \in \mathfrak{A}$. Since $\mathfrak{A}$ is conditionally compact, so is $M\mathfrak{A}$. Thus we

find that for $|x| \leq 1$ the elements of the sequence $\frac{1}{n}(Ax + A^2x + \dots + A^nx)$ belong to the same compact set. Hence we have

$$\frac{1}{n}(Ax + A^2x + \dots + A^nx) \rightarrow Lx \text{ strongly}$$

when $n \rightarrow \infty$. Of course this holds not only for $|x| \leq 1$ but for any x . We have thus obtained the following theorem.

If A is a totally continuous linear operator and if the operators A^n ($n = 1, 2, \dots$) are equi-bounded, then there exists a linear operator L such that

$$\frac{1}{n}(Ax + A^2x + \dots + A^nx) \rightarrow Lx \text{ strongly}$$

when $n \rightarrow \infty$ for any element x .

This is a generalization of a theorem due to M. FRÉCHET¹⁾ on the sequence of iterations of a FREDHOLM kernel.

¹⁾ Loc. cit

Mathematics. — *Generalisations of STEINER's Roman Surface.* By
J. A. BARRAU. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of April 30, 1938).

§ 1. Generalisations of STEINER's Roman surface to space of more dimensions can be obtained in various ways. We shall give two of them by generalising two different classic forms of the parametric equations of that surface¹⁾.

We thus find two series of varieties, each of which shows close analogy to one of two well-known different properties that still occur united in STEINER's surface.

§ 2. We put

$$\left. \begin{aligned} x_1 = \lambda_1^2, \quad x_2 = \lambda_2^2, \quad x_3 = \lambda_3^2, \quad x_4 = \lambda_4^2, \quad x_5 = \lambda_5^2, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0 \end{aligned} \right\}.$$

These equations, where the x_i are homogeneous coordinates in four-dimensional space S_4 , and the λ_i are homogeneous parameters, represent a rational three-dimensional variety V_3^8 of degree eight in S_4 . The points of this variety are in birational relation to those of a parameter-space $\{\lambda_1; \lambda_2; \lambda_3; \lambda_4\}$ where $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$ are taken as fundamental planes, $\lambda_5 = 0$ as unit-plane. That its degree is 8 follows from its intersection with an arbitrary line

$$x_1 = A_4 x_4 + A_5 x_5, \quad x_2 = B_4 x_4 + B_5 x_5, \quad x_3 = C_4 x_4 + C_5 x_5.$$

The only *double points* of the V_3^8 are those corresponding with parameter-points in one of the ten planes:

$$\lambda_i + \lambda_j = 0,$$

e.g. for $i = 1, j = 2$ the two parameter-points

$$\{1; -1; a; b; -(a+b)\} \text{ and } \{-1; 1; a; b; -(a+b)\}$$

give rise to the same point

$$\{x\} \equiv \{1; 1; a^2; b^2; (a+b)^2\}$$

¹⁾ For literature see E. PASCAL, Repertorium der höheren Mathematik, II Geometrie, 334 (1902).

of the V_3^8 . The locus of double points consists of the *ten quadratic cones* C_{ij} :

$$\left. \begin{aligned} x_i &= x_j \\ x_k^2 + x_l^2 + x_m^2 &= 2(x_l x_m + x_k x_m + x_k x_l) \end{aligned} \right\}.$$

Each double cone contains a conic of *pinch-points*, represented by the edge

$$\lambda_i = \lambda_j = 0.$$

Triple points are those corresponding with the triples of edges of trihedral angles

$$\lambda_i + \lambda_j = 0, \quad \lambda_j + \lambda_k = 0, \quad \lambda_i + \lambda_k = 0,$$

e.g. for $i=1, j=2, k=3$ the three parameter-points

$$\{-1; 1; 1; b; -(1+b)\}, \quad \{1; -1; 1; b; -(1+b)\}, \quad \{1; 1; -1; b; -(1+b)\}$$

give rise to the same point

$$\{1; 1; 1; b^2; (1+b)^2\}$$

of the V_3^8 . The locus of triple points consists of the *ten conics* C_{ijk} :

$$\left. \begin{aligned} x_i &= x_j = x_k \\ x_k^2 + x_l^2 + x_m^2 &= 2(x_l x_m + x_k x_m + x_k x_l) \end{aligned} \right\}.$$

As C_{ij} contains C_{ijk} , the cones of double points with the conics of triple points form a configuration Cf(10₃) of DESARGUES¹⁾.

There are five *quadruple points*:

$$\{4; 1; 1; 1; 1\}, \quad \{1; 4; 1; 1; 1\}, \quad \{1; 1; 4; 1; 1\}, \quad \{1; 1; 1; 4; 1\} \\ \text{and } \{1; 1; 1; 1; 4\},$$

the last e.g. arising from the four parameter-points

$$\{-1; 1; 1; 1; -2\}, \quad \{1; -1; 1; 1; -2\}, \quad \{1; 1; -1; 1; -2\} \\ \text{and } \{1; 1; 1; -1; -2\}.$$

Each quadruple point belongs to six double cones and to four triple conics, each cone containing three, each conic two of them.

¹⁾ By projection on ordinary space we get a theorem projectively equivalent with the following:

If A_1, A_2, A_3, A_4, A_5 are five points in general situation, the ten cones with vertices in the middle points of the segments $A_i A_j$ standing on the conics that touch the sides of the triangles $A_k A_l A_m$ in their middle points, form a Cf(10₃) of DESARGUES with ten conics.

The plane section of this configuration is a Cf(10₆, 20₃) of conics and points.

Hyperplane sections of the V_3^8 are obtained by its intersection with a S_3 :

$$A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4 + A_5 x_5 = 0.$$

This V_2^8 corresponds with a quadric in the parameter-space, the discriminant of which is

$$\Delta \equiv \begin{vmatrix} A_1 + A_5 & A_5 & A_5 & A_5 \\ A_5 & A_2 + A_5 & A_5 & A_5 \\ A_5 & A_5 & A_3 + A_5 & A_5 \\ A_5 & A_5 & A_5 & A_4 + A_5 \end{vmatrix}.$$

If the rank of Δ descends to 3, the S_3 is tangent, then

$$\Delta = 0$$

or¹⁾:

$$A_1 A_2 A_3 A_4 + A_1 A_2 A_3 A_5 + A_1 A_2 A_4 A_5 + A_1 A_3 A_4 A_5 + A_2 A_3 A_4 A_5 = 0.$$

Hence this is the class-equation of the V_3^8 . Replacing A_i by u_i , we can write it as

$$\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \frac{1}{u_4} + \frac{1}{u_5} = 0.$$

The class of the V_3^8 is four and it is reciprocal to a V_3^4 :

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} = 0,$$

which has triple points in the fundamental points, double lines in the fundamental edges, while it contains the fundamental planes simply.

The rank of Δ descends to 2 if, and only if, three of the A_i equal zero. If, e.g.,

$$A_1 = A_2 = A_3 = 0,$$

then the hyperplane section with

$$A_4 x_4 + A_5 x_5 = 0$$

corresponds in the parameter-space with

$$A_4 \lambda_4^2 + A_5 \lambda_5^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

¹⁾ A determinant of the order n , in which the elements of the first diagonal are $A_1 + C, A_2 + C, \dots, A_n + C$, and all the other elements are C , contains the term $A_1 A_2 \dots A_n$ and those formed by the product of $(n-1)$ of the A with one factor C , but all the terms in which less than $(n-1)$ factors A would appear have a coefficient zero.

i.e. with two planes harmonical with

$$\lambda_4 = 0 \text{ and } \lambda_5 = 0.$$

As each of these planes represents a surface of fourth degree with one triple point and three double lines passing through it, hence a STEINER's Roman surface, the hyperplane section consists of two such surfaces which have in common a trope-conic (counted twice) and six ordinary conics, bringing the degree of the total intersection to 16, as is due¹⁾.

The rank of Δ descends to one if, and only if, four of the A_i equal zero, the S_3 of intersection is one of the fundamental S_3 , the intersection is a STEINER's Roman surface counted twice and the S_3 is a (higher) trope, of which there are five, reciprocal to the five triple points of V_3^4 .

§ 3. We now put

$$\left. \begin{aligned} x_1 &= 2\lambda_1 \lambda_2, & x_2 &= 2\lambda_1 \lambda_3, & x_3 &= 2\lambda_1 \lambda_4, \\ x_4 &= 2\lambda_2 \lambda_3, & x_5 &= 2\lambda_2 \lambda_4, & x_6 &= 2\lambda_3 \lambda_4, \\ x_7 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \end{aligned} \right\}.$$

These equations, where the x_i are homogeneous coordinates in six-dimensional space S_6 , the λ_i are homogeneous parameters, represent a rational three-dimensional variety V_3^8 of degree eight in S_6 , its points being in birational relation to those of the parameter-space $\{\lambda_1; \lambda_2; \lambda_3; \lambda_4\}$.

That its degree is 8 follows from its intersection with an arbitrary S_3 :

$$\left. \begin{aligned} x_4 &= A_1 x_1 + A_2 x_2 + A_3 x_3 + A_7 x_7, \\ x_5 &= B_1 x_1 + B_2 x_2 + B_3 x_3 + B_7 x_7, \\ x_6 &= C_1 x_1 + C_2 x_2 + C_3 x_3 + C_7 x_7 \end{aligned} \right\}.$$

The only double points of the V_3^8 are those corresponding with parameter-points in one of the six edges of the fundamental tetrahedron, the two parameter-points:

$$\{0; 0; \lambda_3; \lambda_4\} \quad \text{and} \quad \{0; 0; \lambda_4; \lambda_3\}$$

¹⁾ The situation appears clearly in a diagram where $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, $\lambda_4 = 0$ form a regular tetrahedron, $\lambda_5 = 0$ being the plane at infinity. The double planes $\lambda_i + \lambda_j = 0$ are the six faces of a cube and the four planes through the corners of the tetrahedron parallel to the opposite faces. The two planes (1) are parallel to $\lambda_4 = 0$ at equal distances on both sides of it, their line of intersection represents the common trope-conic; their pairs of lines of intersection with the six double planes not passing through the fourth fundamental point $\{0; 0; 0; 1\}$ represent the other six conics.

The question whether the curve of intersection of two Roman surfaces of STEINER can be degenerated into eight different conics, remains, as far as the author knows, unanswered.

e.g. representing the same point

$$\{0; 0; 0; 0; 0; 2\lambda_3\lambda_4; \lambda_3^2 + \lambda_4^2\}$$

of the V_3^8 .

The locus of double points of the V_3^8 consists of the six edges of the fundamental simplex passing through the last corner $X_7 = \{0; 0; 0; 0; 0; 0; 1\}$; the V_3^8 , containing these six edges, cannot be situated in a space of less than six dimensions. The point X_7 is a *quadruple point* of the V_3^8 , as it is represented by the parameter-points

$$\{1; 0; 0; 0\}, \{0; 1; 0; 0\}, \{0; 0; 1; 0\}, \{0; 0; 0; 1\}.$$

Each double line of the V_3^8 contains two *pinch-points*; $X_7 X_6$ e.g. has the pinch-points $\{0; 0; 0; 0; 0; 1; 1\}$ and $\{0; 0; 0; 0; 0; -1; 1\}$, arising from the parameter-points $\{0; 0; 1; 1\}$ and $\{0; 0; -1; 1\}$ resp.

Hyperplane sections of the V_3^8 are obtained by its intersection with a S_5 :

$$A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4 + A_5 x_5 + A_6 x_6 + A_7 x_7 = 0.$$

This V_2^8 corresponds with a quadric in parameter-space, the discriminant of which is

$$\Delta \equiv \begin{vmatrix} A_7 & A_1 & A_2 & A_3 \\ A_1 & A_7 & A_4 & A_5 \\ A_2 & A_4 & A_7 & A_6 \\ A_3 & A_5 & A_6 & A_7 \end{vmatrix}.$$

If the rank of Δ descends to 3, then $\Delta = 0$. Hence this is the class-equation of the locus of hyperplanes S_5 containing tangent- S_3 of the V_3^8 .

If the rank of Δ descends to 2, the hyperplane section degenerates into two V_2^4 , represented by two planes

$$a_1 \lambda_1 + a_2 \lambda_2 + a_3 \lambda_3 + a_4 \lambda_4 = 0$$

and

$$\frac{\lambda_1}{a_1} + \frac{\lambda_2}{a_2} + \frac{\lambda_3}{a_3} + \frac{\lambda_4}{a_4} = 0.$$

The two V_2^4 have in common one conic and six double points on the six double edges; hence such a V_2^4 is not situated in a space of less than five dimensions.

If the rank of Δ descends to one, the two planes, hence also the two V_2^4 , coincide; there are *eight* such *trope-hyperplanes*, represented by the parameter-planes, each counted twice,

$$\{a\} \equiv \{\pm 1; \pm 1; \pm 1; \pm 1\}.$$

A trope-hyperplane intersects each of the six double-edges in one of its two pinch-points.

The projection of the V_3^8 from its quadruple point X_7 on the opposite fundamental S_5 , $X_1 X_2 X_3 X_4 X_5 X_6$, has the parameter-equations:

$$\left. \begin{aligned} x_1 &= \lambda_1 \lambda_2, & x_2 &= \lambda_1 \lambda_3, & x_3 &= \lambda_1 \lambda_4, \\ x_4 &= \lambda_2 \lambda_3, & x_5 &= \lambda_2 \lambda_4, & x_6 &= \lambda_3 \lambda_4 \end{aligned} \right\} \quad (2)$$

hence the parameter-free equations:

$$x_1 x_6 = x_2 x_5 = x_3 x_4.$$

It is a V_3^4 in S_5 , the properties of which can be easily derived from the parameter-representation (2). We see, for instance, that it contains four systems of ∞^2 straight lines, represented by the lines through the corners of the fundamental tetrahedron $\{\lambda\}$, and six systems of ∞^1 quadrics, represented by the planes through the edges of that tetrahedron¹⁾.

§ 4. Reasoning for general n in the same way as was done for $n=3$ in § 2 and § 3, we find that the equations

$$\left. \begin{aligned} x_1 &= \lambda_1^2, \dots, x_i = \lambda_i^2, \dots, x_{n+2} = \lambda_{n+2}^2, \\ \lambda_1 + \lambda_2 + \dots + \lambda + \dots + \lambda_{n+2} &= 0 \end{aligned} \right\}$$

represent a series of rational varieties $V_n^{2^n}$ in S_{n+1} , showing a close analogy to STEINER's Roman surface ($n=2$) as to the property of being reciprocal to a variety V_n^{n+1} in S_{n+1} with n -iple points in the $(n+2)$ corners of the fundamental simplex²⁾, $(n-1)$ -iple lines in the edges, $(n-2)$ -iple planes in the faces, and so on, finally containing the fundamental S_{n-1} simply, as is shown by the intersection of

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_i} + \dots + \frac{1}{x_{n+2}} = 0$$

with the line

$$\lambda A + B \equiv \lambda \{a_1; a_2; \dots; a_k; 0; \dots; 0\} + \{b_1; b_2; \dots; b_{n+2}\}.$$

This gives

$$\frac{1}{\lambda a_1 + b_1} + \frac{1}{\lambda a_2 + b_2} + \dots + \frac{1}{\lambda a_k + b_k} + \frac{1}{b_{k+1}} + \dots + \frac{1}{b_{n+2}} = 0,$$

which is an equation of degree k in λ , giving for every value of k , from 1 till n , k points of intersection not in A , hence $(n+1-k)$ in A .

As a consequence the original $V_n^{2^n}$ has $(n+2)$ trope- S_n , reciprocal

¹⁾ Comp. Curved $(n+1)$ -dimensional varieties containing 2^n systems of straight lines, *Mathematica II*, 161 (1933—1934).

²⁾ See the foot-note on p. 498.

to the n -iple points, the intersection of such a trope- S_n with the $V_n^{2^n}$ being a $V_{n-1}^{2^{n-1}}$ of the same series, counted twice.

On the contrary the equations

$$\left. \begin{aligned} x_1 &= 2\lambda_1\lambda_2, & x_2 &= 2\lambda_1\lambda_3, \dots \\ \cdot & \cdot \cdot \cdot, & x_{\frac{1}{2}n(n+1)} &= 2\lambda_n\lambda_{n+1}, \\ x_{\frac{1}{2}n(n+1)+1} &= \lambda_1^2 + \lambda_2^2 + \dots + \lambda_{n+1}^2 \end{aligned} \right\}$$

represent a series of rational varieties $V_n^{2^n}$ in $S_{\frac{1}{2}n(n+1)}$, showing a close analogy to STEINER's Roman surface ($n=2$) as to the property of possessing one $(n+1)$ -iple point, through which pass $\frac{1}{2}n(n+1)$ double lines, each containing two pinch-points.

Mathematics. — *Ueber Scharen von quadratischen Varietäten.* By
O. BOTTEMA. (Communicated by Prof. W. v. D. WOUDE).

(Communicated at the meeting of April 30, 1938).

Vor einiger Zeit hat DANIELSSON ¹⁾ die folgenden Sätze bewiesen:

1. Alle Schnittkurven der Flächen einer gewöhnlichen Schar von Flächen zweiter Ordnung haben dasselbe Doppelverhältnis.
2. Dieses Doppelverhältnis ist gleich dem Doppelverhältnis des tetraedralen Strahlenkomplexes, zu dem die Erzeugenden der einzelnen Flächen der Schar gehören.

Wir geben im Folgenden Erweiterungen dieser Sätze für R_n .

1. Die Schnitt- V_{n-2}^4 zweier quadratischen Varietäten V_{n-1}^2 einer allgemeinen Schar in R_n sind sämtlich projektiv.

Beweis: Die Exemplare einer Schar (des allgemeinen Typus) von quadratischen Varietäten in R_n können dargestellt werden durch die Gleichungen

$$Q_\lambda \equiv \sum_{i=1}^{n+1} \frac{x_i^2}{a_i + \lambda} = 0,$$

wo λ der Parameter der Schar bedeutet und die Zahlen a_i verschieden sind. Wir betrachten zwei Paare von Varietäten, Q_{λ_1} , Q_{λ_2} und Q_{μ_1} , Q_{μ_2} . Die Nullstellen der Determinante des Büschels $Q_{\lambda_1} - t Q_{\lambda_2} = 0$ sind

$$t_i = \frac{a_i + \lambda_2}{a_i + \lambda_1} \quad (i = 1, 2, \dots, n+1),$$

diejenigen des Büschels $Q_{\mu_1} - \tau Q_{\mu_2} = 0$:

$$\tau_i = \frac{a_i + \mu_2}{a_i + \mu_1} \quad (i = 1, 2, \dots, n+1).$$

Die Paare Q_{λ_1} , Q_{λ_2} und Q_{μ_1} , Q_{μ_2} sind dann und nur dann projektiv, wenn die Zahlen t_i (in beliebiger Anordnung) den Zahlen τ_i proportional sind. Damit die Büschel $(Q_{\lambda_1}, Q_{\lambda_2})$ und (Q_{μ_1}, Q_{μ_2}) projektiv sind — und also auch ihre Basisvarietäten — ist notwendig und hinreichend, dass eine lineare Substitution existiert:

$$\tau = \frac{\alpha t + \beta}{\gamma t + \delta},$$

¹⁾ DANIELSSON, Sätze über Scharen von Flächen zweiter Ordnung, Math. Annalen 109, 521—524 (1934).

welche den Zahlen t_i die Zahlen τ_i zuordnet für $i=1, 2, \dots, n+1$. Eine solche ist aber

$$\tau = \frac{(\mu_2 - \lambda_1) t + (\lambda_2 - \mu_2)}{(\mu_1 - \lambda_1) t + (\lambda_2 - \mu_1)}.$$

Ihre Determinante ist $(\lambda_1 - \lambda_2)(\mu_1 - \mu_2)$ und also von Null verschieden.

Zu jeder Schnitt- V_{n-2}^4 zweier Varietäten der Schar gehört also ein System von $n+1$ Zahlen, z.B. die Zahlen t_i . Die Anordnung dieser Zahlen ist unwesentlich und sie dürfen ausserdem einer linearen Transformation unterworfen werden. Das heisst: *charakteristisch für die allgemeine V_{n-2}^4 ist ein Wurf von n Punkten eines binären Gebietes* und die projektiven Eigenschaften dieses Wurfs bestimmen die projektiven Eigenschaften der Varietät. Eine kanonische Darstellung für die Koordinaten der Punkte des Wurfs erhält man z.B. für $\lambda_1 = \infty, \lambda_2 = 0$, nämlich

$$t_i = a_i.$$

Der zweite von DANIELSSON gegebene Satz sagt aus, dass für $n=3$ der für die Schnittkurve charakteristische Wurf projektiv ist mit der aus vier Punkten bestehenden Figur, welche das Fundamentaltetraeder der Schar ausschneidet auf einer beliebigen Erzeugenden. Da es in R_n auf die Varietäten einer Schar sämtlich ∞^{2n-4} Gerade gibt, während nur ∞^{n-1} Gerade durch das Fundamentalsimplex geschnitten werden in einer Gruppe von n Punkten, welche mit einem gegebenen Wurf projektiv ist, wird eine Erweiterung des Satzes in der gegebenen Fassung nicht möglich sein. Durch die Bemerkung, dass im Falle $n=3$ auch die *Tangenten* der Schnittkurven V_1^4 zum tetraedralen Komplex gehören, werden wir zum folgenden Satz geführt:

II. Die $n+1$ linearen R_{n-1} , welche durch den berührenden R_{n-2} in einem beliebigen Punkt der V_{n-2}^4 und je einen Eckpunkt des Fundamentalsimplexes gehen, bilden einen Wurf, welcher mit dem charakteristischen Wurf der V_{n-2}^4 (und der ganzen Schar) projektiv ist.

Beweis: Es sei die V_{n-2}^4 die Schnittvarietät der quadratischen Varietäten $Q_{\lambda_1} = 0$ und $Q_{\lambda_2} = 0$. Der berührende R_{n-2} in dem Punkt $P \equiv x_1^0, x_2^0 \dots x_n^0$ ist der Durchschnitt der linearen R_{n-1} mit den Gleichungen:

$$G_1 \equiv \sum \frac{x_i^0 x_i}{a_i + \lambda_1} = 0, \quad G_2 \equiv \sum \frac{x_i^0 x_i}{a_i + \lambda_2} = 0.$$

Die Gleichungen der fraglichen R_{n-1} lauten also

$$G_1 - k G_2 = 0;$$

für denjenigen, welcher durch den Eckpunkt $x_i = 0$ ($i \neq p$) geht, findet man

$$k = k_p = \frac{a_p + \lambda_2}{a_p + \lambda_1},$$

woraus der Satz unmittelbar hervorgeht.

Meteorology. — *On a period of 5.25 years in rainfall, temperature and pressure.* By H. J. DE BOER. (Communicated by Prof. E. VAN EVERDINGEN.)

(Communicated at the meeting of April 30, 1938).

A periodicity of just over five years in the North and East component of wind direction and in the airpressure over the British Isles, investigated by BROOKS¹⁾, and the fact that the winters of 1924, 1929 and 1934 were cold in the Netherlands²⁾ led to the detection and investigation of a period of 5.25 years in and around the region of the Northern Atlantic Ocean.

Suppose we compare the departures from normal of the seasonal means of some meteorological element during two different series of the same length in years. If there is periodicity in the deviations, the positive correlation between the corresponding values in two series of years will be maximum, when the difference of time Δ between the beginning of the two series is an even multiple of half the period; the negative correlation will be maximum when the difference of time Δ is an odd multiple of half the period.

For the determination of the period we take successive values of Δ and calculate for each value the correlation coefficient. So we get in the curve for the correlation coefficients many maxima and minima indicating even and odd multiples of half the period respectively. The distance in time between two successive maxima or minima is of the order of the length of one period. Thus we can easily compute how many periods correspond to each maximum. From each maximum and each minimum we can calculate the length of the period. Then we may consider the arithmetic mean of all the calculated lengths to be the length of the supposed period.

In order to avoid the elaborate computation of the correlation coefficients, we introduce a new quantity, "the sign coefficient". When comparing two series of departures from the mean, we note a plus sign for two corresponding deviations of the same sign. When the corresponding deviations are opposite in sign, we note a minus sign. We note the value 0 when at least one of the deviations is zero. Now the sign coefficient is the quotient obtained when the difference between the number of plus signs and the number of minus signs is divided by the total number of figures, the zero values included.

¹⁾ C. E. P. BROOKS, the *Meteorological Magazine*, **72**, 204 (1937).

²⁾ S. W. VISSER, *Hemel en Dampkring*, **36**, 13 (1938).

Dr. S. W. VISSER has calculated the correlation between 50 sign coefficients in the interval 1901—1937 and the 50 corresponding correlation coefficients. The correlation coefficient proved to be 0.82.

This shows that we may use the sign coefficient as a good approximation of the correlation coefficient, except when the deviations are very small.

The above-mentioned scheme has been applied at first to the seasonal temperature deviations at De Bilt for the interval from 1849 to 1938. We have only used the winter deviations after elimination of the 27-month period³⁾ and have started from a value of Δ of 20 years; then we have taken Δ equal to 21 years and so on up to 70 years. Thus we have found maxima at Δ values of 21.0, 25.4, 31.5, 36.6, 42.9, 47.2, 53.2, 63.7 and 67.7 years, evidently corresponding to 4, 5, 6, 7, 8, 9, 10, 12 and 13 periods. From these maxima, the mean length of the period has been calculated to be 5.25 years. The same result we deduced from the minima, which we found at 23.8, 27.7, 34.7, 40.0, 44.7, 49.5, 55.4 and 65.2 years, corresponding to $4\frac{1}{2}$, $5\frac{1}{2}$, $6\frac{1}{2}$, $7\frac{1}{2}$, $8\frac{1}{2}$, $9\frac{1}{2}$, $10\frac{1}{2}$ and $12\frac{1}{2}$ periods. These data too gave us a mean length of the period of 5.25 years. (See figure 1.)

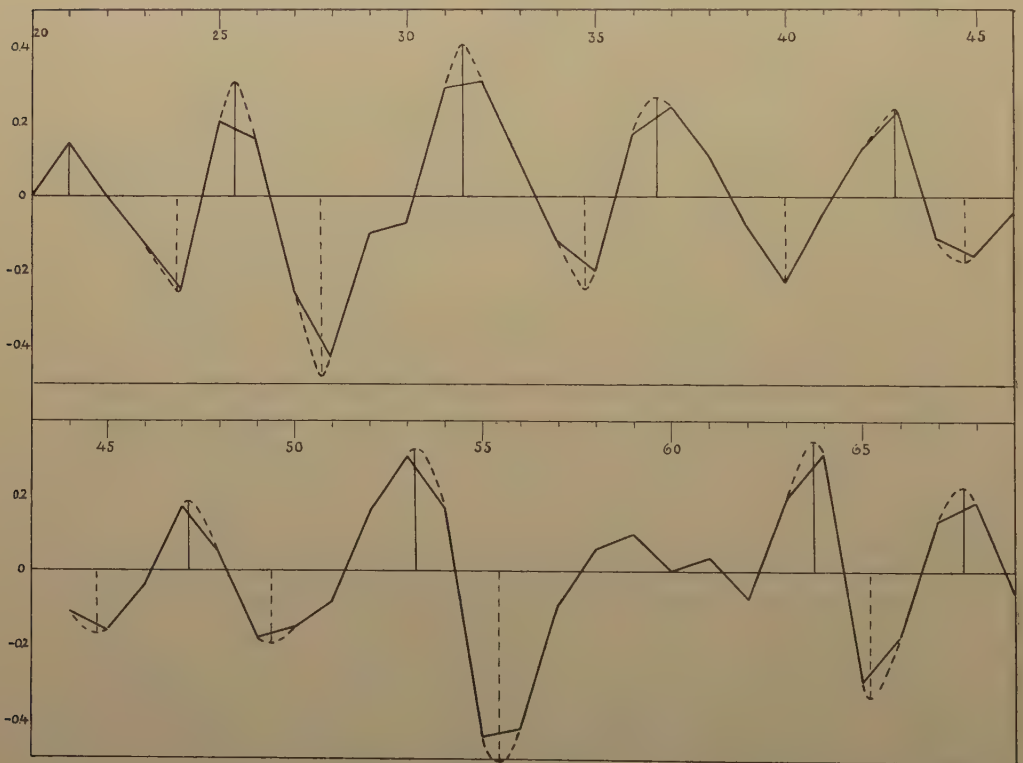


Fig. 1. Sign coefficients of temperature at De Bilt.

In order to show the stability of the 5.25-year period in the temperature

³⁾ S. W. VISSER, Proc. Royal Acad. Amsterdam, 40, 513 (1937).

of De Bilt, the deviations for the four seasons corrected for the 27-month period have been arranged in cycles of 21 years duration and we have determined the mean for each cycle. The stability of the maximum during the interval from 1849 to 1938 is satisfactory (table 1).

TABLE 1. Stability of the 5.25 year period in the temperature of De Bilt (degrees C.)

1849—1869	{	0.8	-0.6	-0.1	-0.9	-0.8	0.4	-0.3	-0.3	0.3	0.4	-0.2
		0.4	0.4	0.5	1.4	1.0	0.0	0.4	-0.3	-0.0	-0.4	
1870—1890	{	0.2	-0.1	-0.4	-0.3	-0.4	-0.1	-0.1	0.7	-0.8	0.1	0.1
		0.1	0.6	-0.0	1.0	-0.1	-0.1	-0.0	0.5	-0.8	-0.2	
1891—1911	{	-0.5	-0.3	-0.6	-0.8	-0.0	-0.2	-0.1	0.4	-0.8	0.4	0.2
		0.0	-0.0	0.8	0.4	-0.2	-0.8	-0.1	0.6	0.3	0.0	
1912—1932	{	-0.2	0.1	0.3	-0.3	0.6	-1.1	-1.4	0.3	0.1	0.4	0.2
		0.8	1.3	-0.7	-0.4	0.1	0.8	0.5	-0.3	0.2	-0.6	
1849—1937	{	0.2	-0.1	-0.1	-0.5	-0.2	-0.3	-0.5	0.4	-0.1	0.3	0.1
		0.4	0.6	0.1	0.6	0.2	0.0	0.2	0.1	-0.1	-0.3	

It seemed worth while to apply this method also to precipitation at De Bilt. We used for precipitation the four seasonal deviations after elimination of the 27-month period and we took values for Δ of 20, $20\frac{1}{4}$, $20\frac{1}{2}$, $20\frac{3}{4}$, and so on up to 70 years. Again we found a period of 5.25 years. Here the stability of the minima in the various 21 season cycles is satisfactory, though to a smaller degree than with the maximum in the case of temperature during the interval from 1849 to 1938 (table 2).

TABLE 2. Stability of the 5.25 year period in rainfall of De Bilt (mm).

1849—1869	{	-37.1	10.9	-29.6	42.5	-7.8	-2.2	-26.5	15.7	0.9	21.0	-11.1
		-17.8	3.4	-28.9	-27.7	-17.0	-30.8	7.0	-6.9	-33.8	-1.2	
1870—1890	{	6.8	4.5	25.6	13.2	11.3	37.7	6.8	4.1	24.7	12.1	13.0
		16.5	21.7	-1.0	13.0	7.2	-13.6	-5.0	-17.8	1.0	-44.5	
1891—1911	{	-43.6	15.2	20.1	-32.1	10.1	-38.0	-16.7	50.6	-4.1	-11.7	-19.5
		22.0	-10.5	-18.2	20.1	18.4	-0.5	8.1	-2.0	-13.5	-13.7	
1912—1932	{	-3.7	32.2	27.6	21.0	-3.3	-10.0	-4.4	-0.6	9.0	7.4	-1.8
		37.2	56.5	40.6	-16.0	10.5	-2.2	-27.3	-10.5	-14.1	39.8	
1849—1937	{	-23.1	13.9	9.8	9.2	-1.0	-20.3	-12.5	15.8	9.2	7.8	-6.1
		15.6	21.1	-3.0	-1.4	7.2	-5.3	-1.9	-10.5	-17.8	-5.0	

Following VISSER's example in the case of the 27-month period in the rainfall, we have investigated the character of the 5.25 year period on and around the Northern Atlantic Ocean. We examined the rainfall for most of the stations from 1876 to 1929 inclusive. The present period of 5.25 years started with the spring of 1938 (March 1st 1938). Harmonic analysis was applied in order to detect some general features of this period. We only used the first two terms of the harmonic analysis.

We found two important features (see table 3):

1. a phase difference combined with a decrease of amplitude on the Atlantic Ocean in the direction of the Gulfstream;
2. a retardation at Thorshavn.

TABLE 3. Harmonic analysis of 5.25-year period on the Atlantic Ocean.

								Maxi- mum in season
1.	Ceara	3.7° N	38.5° W	44.2	$\sin(x + 125^\circ) + 28.6 \sin(2x + 59^\circ)$	20.0		
2.	{ Paramaribo	5.8 N	55.3 W	47.6	43	24.0	358	3.7
	{ Barbados	13.1 N	59.6 W	21.6	42	15.7	321	3.8
3.	Bermuda	32.3 N	64.8 W	14.2	305	16.4	344	9.5
4.	Ponta Delgada	37.7 N	25.7 W	15.7	154	5.9	284	18.3
5.	{ Valentia	51.9 N	10.2 W	15.4	308	7.6	291	9.3
	{ Netherlands	52.1 N	5.2 E	7.3	282	5.8	34	10.8
6.	Stykkisholm	65.5 N	22.8 W	10.2	156	7.7	51	18.2
7.	Thorshavn	62.0 N	6.8 W	9.0	30	9.4	224	4.5
8.	Bodö	67.3 N	14.4 E	16.5	75	11.1	154	1.6

Even if we take into account the large yearly rainfall in the neighbourhood of Ceara, the amplitude in this region is larger, which may be explained by the supposition that here the disturbances take their origin. The phase differences are not inconsistent with the hypothesis that the disturbances take a little over eleven years to cross the Ocean from Ceara to Bodö. The travel-time for the 27-month period over the Ocean is six or seven months. So the proportion of the first-mentioned travel-time to the second is as 20 : 1 or the proportion of their velocities is 1 : 20. It is a known fact that the normal proportion of current velocity to wind-velocity in the open Ocean is 1 : 20.

We have calculated from the "Oceanographische en Meteorologische waarnemingen in den Atlantischen Oceaan, Koninklijk Nederlandsch Meteorologisch Instituut, Nr 110" the yearly average of the component of the current-velocity along the direction from Bermuda to Valentia. The yearly average proved to be about 6.8 km in 24 hours. As follows from table 3, the transmission-velocity of the 5.25 year period from Bermuda to Valentia, being 5800 km in 5 years, proved to be 3.2 km in 24 hours.

Also the wind-velocity, averaged over a year, at ships-height above sea-level in the direction of the Gulfstream from Bermuda to Valentia has been calculated. This velocity proved to be 105 km in 24 hours. The

travel-time of the 27-month period from Bermuda to Valentia is 2.2 months. Hence the transmission-velocity is about 86 km in 24 hours.

The proportion between the yearly average of the current-velocity and the wind-velocity at ships-height above sea-level is 1 : 16, whilst the proportion between the transmission-velocity of the 5.25-year period and the transmission-velocity of the 27-month period is 1 : 26. Considering that the proportion 1 : 20 is only applicable to regions with constant wind direction, such as trade winds, and the real path of the water masses will probably be more curved than that of the air masses, we may consider the result of this rough calculation as satisfactory.

These facts might be explained by supposing that departures from the normal, originating in the neighbourhood of Ceara and showing a periodicity of $5\frac{1}{4}$ years, are transmitted by the water of the Gulfstream, and that other departures with a periodicity of $2\frac{1}{4}$ years are transmitted by the general circulation of air over the Ocean.

VISSER has computed the correlation coefficients for temperature and precipitation at De Bilt between the values according to the 5.25-year period, calculated from the data in the interval 1849—1937 and the observed data in the interval 1901—1937. The same has been done for the 27-month period (table 4).

TABLE 4. Correlation coefficients between observed and calculated values for the 5.25-year and the 27-month periods.

	Precipitation				Temperature			
	Wi	Spr.	Su	Au	Wi	Spr.	Su	Au
27 month period	+0.020	+0.282	+0.215	+0.323	+0.120	+0.253	-0.013	+0.129
5.25 year period	+0.381	+0.290	+0.177	+0.218	+0.415	+0.195	+0.280	+0.197

Especially for temperature the correlation coefficients for the $5\frac{1}{4}$ year cycle are larger and more stable. This is due to the fact that water is less subject to disturbing influences than air.

Now we will come back to the indication of a retardation at Thorshavn. VISSER found the same result for the 27-month period. The small number of stations available does not permit any definite conclusion, but if the retardation is real in both cases we would be led to the supposition that Thorshavn is not on the mean path of the general circulation and of the Gulfstream between Valencia and Bodö, and that the disturbances reach it only on some return track after intermixing with cold currents from the North.

C. E. P. BROOKS ⁴⁾ has found a period of just over five years in airpressure and in the North and East component of wind direction. We will repeat what BROOKS has written about the period of 5.05 years in airpressure and wind direction over the British Isles:

⁴⁾ C. E. P. BROOKS, l.c.

"At Edinburgh the pressure data for the years from 1772—1816 and 1907—1936 showed no evidence of a periodicity of about five years, but the period 1817—1906 showed a periodicity of 5.06 years very clearly." About the *N* component of wind direction "At London it was 5.09 years from 1715 to 1797, 5.03 years from 1802 to 1887 and again 5.09 years from 1887 onwards. At Edinburgh there were breaks about 1840 and 1860, while at Dublin the length was about 5.0 years from 1726 to 1765 and 5.1 years from 1831 to 1936."

Hence BROOKS has found periods of various lengths at stations at relatively small distances. Moreover, the amplitude of the periodicity was different at various stations. Therefore we have studied the 27-month period and the 5.25 year period in airpressure and in temperature at Edinburgh. We have compared the results with those we found in temperature at Zwanenburg and at De Bilt.

In the first place there was a break about 1842—1843 in the two meteorological elements at the stations mentioned for the 5.25-year period as well as for the 27-month period.

In the second place the two periods were much better developed (i.e. the proportion of the amplitude of the first term to that of the second term of the harmonic analysis is markedly larger after the break, than before). In the third place the two periods in temperature have been better developed in the neighbourhood of the Gulfstream than on the continent, though the amplitude of the first term on the continent is larger than on the coast of the Atlantic Ocean.

Lastly the phase differences in the three cases between the intervals at both sides of the break are in satisfactory mutual agreement except for the 27-month period in airpressure at Edinburgh. Those facts are shown by table 5 (I is the interval 1770—summer 1843, II is the interval autumn 1843—1930, III is the interval 1770—1842 and IV 1843—1930).

TABLE 5. Comparison of harmonic analysis before and after the break $\pm$ 1843.

5.25 year period		27 month period	
Airpressure Edinburgh			
I	$0.22 \sin(x + 306^\circ) + 0.23 \sin(2x + 227^\circ)$	III	$0.35 \sin(x + 73^\circ) + 0.29 \sin(2x + 49^\circ)$
II	$0.51 \sin(x + 98^\circ) + 0.07 \sin(2x + 278^\circ)$	IV	$0.37 \sin(x + 190^\circ) + 0.49 \sin(2x + 198^\circ)$
Temperature Edinburgh			
I	$0.09 \sin(x + 108^\circ) + 0.07 \sin(2x + 289^\circ)$	III	$0.07 \sin(x + 95^\circ) + 0.10 \sin(2x + 26^\circ)$
II	$0.23 \sin(x + 215^\circ) + 0.06 \sin(2x + 150^\circ)$	IV	$0.21 \sin(x + 135^\circ) + 0.09 \sin(2x + 40^\circ)$
Temperature De Bilt and Zwanenburg			
I	$0.20 \sin(x + 81^\circ) + 0.17 \sin(2x + 178^\circ)$	III	$0.12 \sin(x + 188^\circ) + 0.23 \sin(2x + 37^\circ)$
II	$0.33 \sin(x + 229^\circ) + 0.10 \sin(2x + 78^\circ)$	IV	$0.11 \sin(x + 219^\circ) + 0.12 \sin(2x + 236^\circ)$

In order to show more clearly the break about 1842 which BROOKS has already mentioned, we will take one of the numerical tables of our investigation, namely the 27-month period in airpressure at Edinburgh (table 6).

TABLE 6. The break in the airpressure at Edinburgh.

1770—1779	— 0.3	0.3	—10.7	— 6.4	— 1.1	1.0	—14.1	0.9	—11.3
1779—1788	2.5	— 3.9	14.9	0.8	— 4.4	— 0.3	9.2	3.1	5.9
1788—1797	1.5	1.2	— 0.6	0.2	— 4.9	— 8.1	— 2.0	— 9.2	0.2
1797—1806	10.9	— 8.8	—15.7	3.3	— 6.0	6.1	— 7.1	— 6.2	11.4
1806—1815	— 6.8	8.2	— 5.4	— 2.0	— 1.7	4.6	— 0.3	— 3.8	4.0
1815—1824	— 2.9	— 8.3	10.8	— 5.3	— 9.8	1.3	4.6	—11.2	1.0
1824—1833	0.9	1.8	5.5	— 1.2	— 6.3	9.1	11.9	— 5.9	— 4.9
1833—1842	0.7	— 1.7	6.5	— 4.2	—12.7	9.8	— 7.8	4.1	0.5
1842—1851	3.1	— 2.9	1.6	3.1	1.4	— 2.7	— 6.7	— 8.2	— 6.3
1851—1860	2.1	5.8	— 6.6	9.7	4.7	9.9	— 7.5	— 0.1	—11.1
1860—1869	— 1.5	— 5.5	— 4.8	5.4	6.9	3.4	—11.3	4.3	1.7
1869—1878	— 5.1	4.8	9.2	— 1.9	1.1	— 4.0	— 2.9	— 2.7	3.2
1878—1887	12.0	— 7.6	— 4.5	5.4	— 2.3	— 1.5	— 1.6	11.9	7.8
1887—1896	— 3.3	— 1.8	4.1	8.5	6.7	— 3.0	0.3	19.6	4.5
1896—1905	10.5	6.2	0.2	— 3.1	3.4	— 2.9	— 0.8	5.4	10.5
1905—1914	— 0.7	— 7.9	— 8.4	— 1.6	1.4	3.1	6.7	13.0	7.0
1914—1923	— 0.2	— 0.8	— 4.6	7.7	— 2.9	2.1	— 7.3	1.6	6.8
1923—1930	—11.6	— 2.4	— 0.2	—14.3	— 1.6	— 2.8	4.0	9.0	— 5.3

Table 6 contains 18 cycles of nine years duration, each cycle containing the sum of four periods of nine seasons. The feature of the deviations before and after the break is remarkably different.

The mean deviations of the temperature at Edinburgh and at Zwanenburg—De Bilt before and after the break are in very good agreement for the two periods. We will show this in the following table.

TABLE 7. Comparison of the deviations of the temperature at E. and Zw. — De B.

The 5.25-year period at Edinburg and Zwanenburg 1770 — autumn 1843

E. 0.0	—0.4	0.1	—0.2	0.3	—0.0	—0.0	0.0	—0.4	0.2	—0.1
Zw. 0.2	—0.4	0.1	—0.2	0.6	—0.0	—0.1	0.1	—0.5	0.3	—0.2
E. 0.5	—0.0	—0.5	0.1	0.2	0.1	—0.1	—0.2	0.4	0.3	
Zw. 0.3	—0.5	—0.7	—0.3	0.1	—0.0	0.1	—0.0	0.6	0.5	

TABLE 7. (Continued).

The 5.25-year period at Edinburgh and at De Bilt autumn 1843—1930										
E. 0.2	—0.0	—0.5	—0.4	—0.3	—0.2	—0.1	—0.1	—0.0	0.2	0.2
d.B. 0.2	—0.1	—0.1	—0.5	—0.2	—0.3	—0.5	0.4	—0.1	0.3	0.1
E. 0.0	0.2	0.0	0.6	0.1	0.0	0.0	0.2	0.1	—0.2	
d.B. 0.4	0.6	0.1	0.6	0.2	0.0	0.2	0.1	—0.1	—0.3	
The 27-month period at Edinburgh and at Zwanaenburg 1770—1842										
E. 0.2	0.1	—0.1	—0.0	—0.2	0.0	0.1	—0.1	—0.1	—0.1	
Zw. —0.1	0.2	—0.4	—0.3	0.2	0.2	0.3	—0.0	—0.0	—0.3	
The 27-month period at Edingburgh and at De Bilt 1842—1930										
E. 0.3	—0.0	—0.1	—0.3	—0.2	0.0	0.1	0.1	0.1	0.1	
d.B. —0.1	—0.3	—0.0	0.1	—0.1	0.0	0.2	0.0	0.0	0.1	

Botany. — *Influence of hydrogen-ion concentration on the growth rate of the Avena coleoptile.* (Preliminary note). By ANNA M. A. VAN SANTEN. (Communicated by Prof. G. VAN ITERSON JR.).

(Communicated at the meeting of April 30, 1938).

Our knowledge of the influence of pH on growth is still far from satisfactory, although WENT and THIMANN (1937) in their book on *Phytohormones* seem to regard the subject as closed.

STRUGGER (1932, 1933, 1934) working with roots and hypocotyls of *Helianthus* finds in both cases a two-peaked curve for the variation of growth with pH. The peaks lie approximately at pH 4.5 and 5.5, between them a minimum at pH 5.1. This curve closely resembles the two-peaked curve for the variation of protoplasm viscosity with pH.

STRUGGER considers an increase in viscosity of the protoplasm as the primary cause of growth. It would lead to a "not osmotic increase of inner pressure" and this to stretching of the cell wall.

BONNER (1934), experimenting with coleoptiles of *Avena*, confirms the occurrence of STRUGGER's "acid curvatures". When a narrow strip of epidermis is removed on one side to allow an acid buffer to enter, the coleoptiles curve away from the wound just as did the *Helianthus* hypocotyls.

BONNER's results, however, differ widely from those of STRUGGER. Instead of a two-peaked curve he found a curve indicating that growth increases with increase of hydrogen-ion concentration. This curve is an S-curve resembling the titration curve of an acid.

BONNER compares it with the titration curve of "the weak acid, growth substance" and finds they match closely. Here he is, however, rather optimistic.

In the first place his curve as originally found does not fit the titration curve at all. He therefore argues that we ought not to consider the variation of growth in relation to the pH of the medium, but rather to the pH of the cell sap. When he applies this as a correction, the growth curve matches the titration curve fairly well. However the values he uses as "internal pH of the cell" or "actual acidity" are determined by "measuring colorimetrically the pH of the crushed coleoptiles". This hardly seems permissible.

Furthermore the titration curve of "the weak acid, growth substance" appears to be that of hetero-auxin with a $pK=4.75$. BONNER could not yet know the pK of auxin, since it was determined for the first time in 1935 by KOSTERMANS to be 5.0. It is regrettable that WENT and THIMANN

in their recent monograph on phytohormones cite BONNER's experiments even using the term auxin instead of growth substance. Hereby the error is furthered and continued. For if we compare BONNER's growth curve with the titration curve of auxin instead of that of hetero-auxin, it is definitely not possible to superimpose both curves.

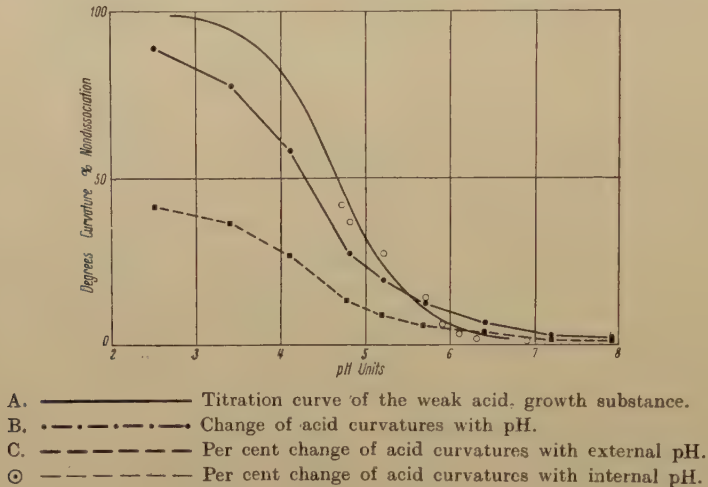


Fig. 1. (after BONNER 1934.)

So the present aspect of the problem is not satisfactory.

On the one hand we have BONNER's theory that in *Avena* acid acts on cell stretching through its influence on growth substance only. By adding hydrogen-ions the dissociation of this weak acid is reduced and the undissociated molecule is the active form.

This theory, accepted by some American workers on growth as universally valid, fails however if the two curves do not exactly match. BONNER himself stresses the point that his data apply only to *Avena* and that the facts may be quite different for *Helianthus* and other plants.

Lately doubt has arisen in several quarters as to the general validity of this dissociation hypothesis. MEESTERS (1936) and LANE (1936) reported independently that the growth of roots is not influenced by changes in the pH of the culture solution, whereas the conversion of auxin from the salt form into the free acid might be expected to result in an inhibition of root growth such as was obtained with 3-indole acetic acid.

On the other hand STRUGGER's two-peaked curves are cited again and again, even in very recent papers (AMLONG 1936, MEESTERS 1936, BORRIS 1937, WUHRMANN 1937, a.o.) and are accepted by many authors on growth and protoplasmic problems in Europe.

It therefore seems worthwhile to publish a few preliminary results on this subject.

Material and method.

A possible cause for the discrepancy in BONNER's results is the lack of a method delicate enough to satisfactorily measure small differences in "acid growth". He states that "the curves upon the basic side (of pH 6.1) are so small that it is impossible to draw any conclusions".

Therefore a microscopic method might here be preferable to a macroscopic one.

STRUGGER measured acid growth with a horizontal microscope. I repeated his experiments and found the method a cumbrous and not very accurate one on account of the large variability of the material. Data about this will follow later.

A more promising technique was the one first used by BONNER (1933) and afterwards by several others. BONNER cut small sections from *Avena* coleoptiles, put these hollow cylinders on thin glass rods in different solutions and measured the growth rate with a microscope with eyepiece micrometer.

With this method, however, we can only use hollow organs for our experiments. And if we want to solve the problem of the relation of pH to growth as a whole, it would be necessary to consider the growth rate of roots as well as that of stems from different plants under the same conditions.

Besides it became apparent that this way of fixing the sections may lead to rather serious mistakes.

The inner diameter of the hollow coleoptiles vary in width ever so slightly. But even so the widest ones are apt to turn round on the glass rods if the solution in which they are lying is stirred.

The difference in length between two opposite sides of a section amounted sometimes to 60μ in my earlier experiments, whereas the growth per hour may be 20μ and less. To avoid this rotation it was necessary to move the solution as little as possible.

On the other hand any organ lying in a not very strongly buffered solution will change the pH of its immediate surroundings. As we want on the contrary to change the pH of the well buffered cell contents and may not use strongly buffered media for fear of damaging the protoplasm the only way is to stir the solution very thoroughly indeed.

I therefore changed BONNER's method in the following way.

A very narrow (0.5 mm wide) strip of pure white vaseline was fixed on a slide. By means of a fine painters brush sections were mounted on this strip in such a way, that only a small area in the middle of the section came into contact with the vaseline.

The growth of these sections was measured in a shallow layer of solution in a petri dish by means of a Zeiss microscope with eyepiece micrometer (enlargement $55\times$). During the experiment the solution was stirred and aerated by means of an air flow from a capillary tube.

The sections were kept in complete darkness throughout the experiment. Orange light (Schott filter O.G. 2) was used for measuring.

In this paper I want to present only the results obtained with *Avena* coleoptiles.

The oat seedlings were grown in a darkroom at 23° C and 90 % humidity. When they were three days old and measured about 17 mm, I cut sections from them by means of the coleoptile microtome after VAN DER WEY (1932). This instrument enabled me to cut zones from ten coleoptiles at a time at every desired distance from the tip and to every desired length with great accuracy. Ten such sections formed a series and measured f.i. $1491 \pm 10 \mu$, the next cut series $1459 \pm 10 \mu$. In every experiment two series were used for each different pH.

According to AVERY and BURKHOLDER (1936) the zone of maximum growth is situated from 6 to 8 mm from the tip, when the coleoptiles are from 14 to 22 mm long (about three days old at 23° C).

As I took the cylinder from 6 to 7.5 mm from the tip, I could be sure of observing cells that were all in the same stage of growth.

BONNER cut his sections from four days old plants. But at this age the coleoptiles are near the end of their growth period and the growth rate is much less than the day before.

Only coleoptiles which had practically the same length were used for each experiment.

As a buffer solution a 0.01 M mixture of KH_2PO_4 and Na_2HPO_4 was used. To obtain buffers on the acid side of pH 5.4 small amounts of 0.1 N H_3PO_4 were added, on the alkaline side of pH 7.5 a 0.1 N solution of Na_3PO_4 served the same end. The water had been distilled over glass and only purest salts were used. The pH was measured by means of a quinhydrone electrode.

I could not use either MICHAELIS' acetate buffers or MC ILVAINE's phosphate-citric acid buffers, used respectively by STRUGGER and BONNER as I had previously found them to be toxic to roots of *Helianthus* after a few hours.

Experimental and discussion.

Can acid promote growth in cells that are free of auxin?

As yet there is no answer to this question, since BONNER's coleoptiles still contained growth substance.

Still this is a very important question, since it concerns the validity of WENT's axiom: no growth without growth substance (WENT 1928).

My first object was therefore to try to obtain coleoptile sections that were free of auxin.

Sections cut from plants grown under normal conditions continued growing in distilled water for three to four days. This is not surprising

since THIMANN (1934) clearly proved that every coleoptile cell contains auxin, although in smaller amounts than the auxin producing tip.

Decapitation made no difference to speak of in the growth rate, as may be seen in figure 2.

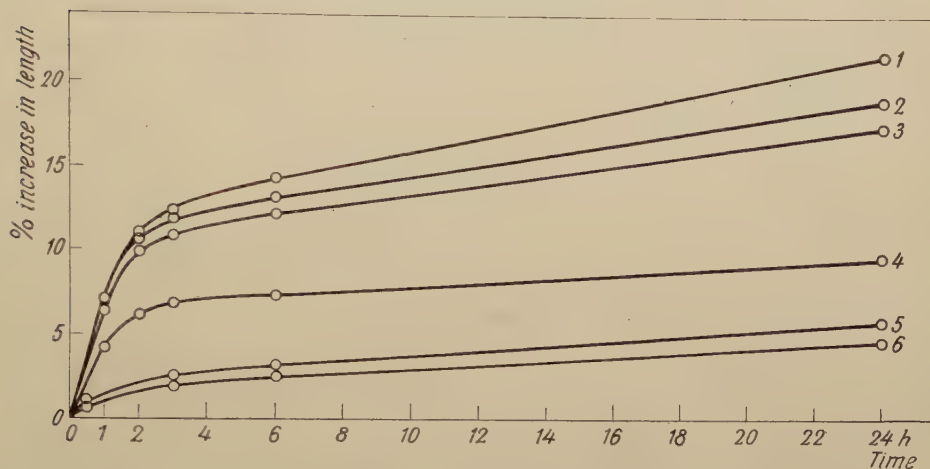


Fig. 2. Growth of sections from variously treated plants in distilled water.

1. Three days old. Control.
2. idem Decapitated twice, with primary leaf.
3. idem idem , without primary leaf.
4. idem Deseeded 18 hours previously. Decapitated.
5. Four days old. Deseeded 24 hours previously. Not decapitated.
6. idem idem . Decapitated.

Curve 1 represents the growth of sections from normal three days old coleoptiles as a control. 2 and 3 concern plants which had been decapitated twice, resp. four and two hours previously. Curve 2 refers to cylinders still containing the primary leaf, 3 to cylinders from which the primary leaf had been removed.

It is clear that the presence of the primary leaf does not affect the growth rate.

Moreover the variability between the individual sections of one series was larger when the primary leaf was lacking. This was probably due to an ever so slight injury of the cells. I therefore always used the cylinders without depriving them of the primary leaf.

SKOOG (1937) could obtain coleoptiles that were practically auxin-less in the standard *Avena* test by deseeding the plants. By removing the source of the precursor of auxin, the seed, he prevented coleoptiles to regenerate growth substance.

Therefore I deseeded plants after SKOOG's technique 18 to 24 hours before cutting the sections. The results may also be seen in fig. 2. Curve 4 represents the growth of cylinders from three days old plants, which had

been deseeded 18 hours previously and decapitated 110 minutes before cutting the sections.

TABLE I. Variation of growth with pH

pH	% growth in 24 h.	number of sections	pH	% growth in 24 h.	number of sections
3.42	— 0.95	18	5.82	12.3	20
3.75	14.1	10	6.10	12.3	28
3.83	20.7	20	6.28	12.6	57
3.95	19.4	28	6.43	12.4	30
4.13	21.3	40	6.53	12.7	30
4.18	23.9	40	6.67	11.9	17
4.24	23.2	20	6.95	11.5	217
4.53	21.3	20	7.00	10.66	153
5.06	17.4	20	7.08	12.3	20
5.33	16.6	20	7.56	11.5	38
5.45	16.2	19	7.65	9.7	19
5.69	13.6	20	7.97	9.4	19

It is evident that the growth rate is reduced, but that there is still a considerable increase in length.

Curves 5 and 6 refer to sections from four days old plants. Normally these sections grow already at a much lower rate than sections from plants that are three days old. But even in those seedlings growth is not prevented altogether by deseeding and decapitation.

I therefore had to assume that it is impossible to obtain cells that are absolutely free of growth substance by means of any technique we are acquainted with up till now.

As the difference in growth rate of the variously treated sections is only a relative one, I only used normal plants for the experiments in buffer solutions of various pH. In this way one can work more quickly and thus use more and larger series for each experiment.

The results of a great number of experiments at different pH values are summarized in table I.

I compared the percent increase in length after 24 hours in each different buffer solution. After such a long time we may reasonably assume that the hydrogen-ion concentration of the cell contents and that of the buffer solution have reached a stage of equilibrium, at least if there is a constant flow of fresh solution round the sections, as was the case here. The correctness of this assumption, however, becomes apparent when we

consider how closely the growth values for each different pH follow the titration curve of auxin (figure 3).

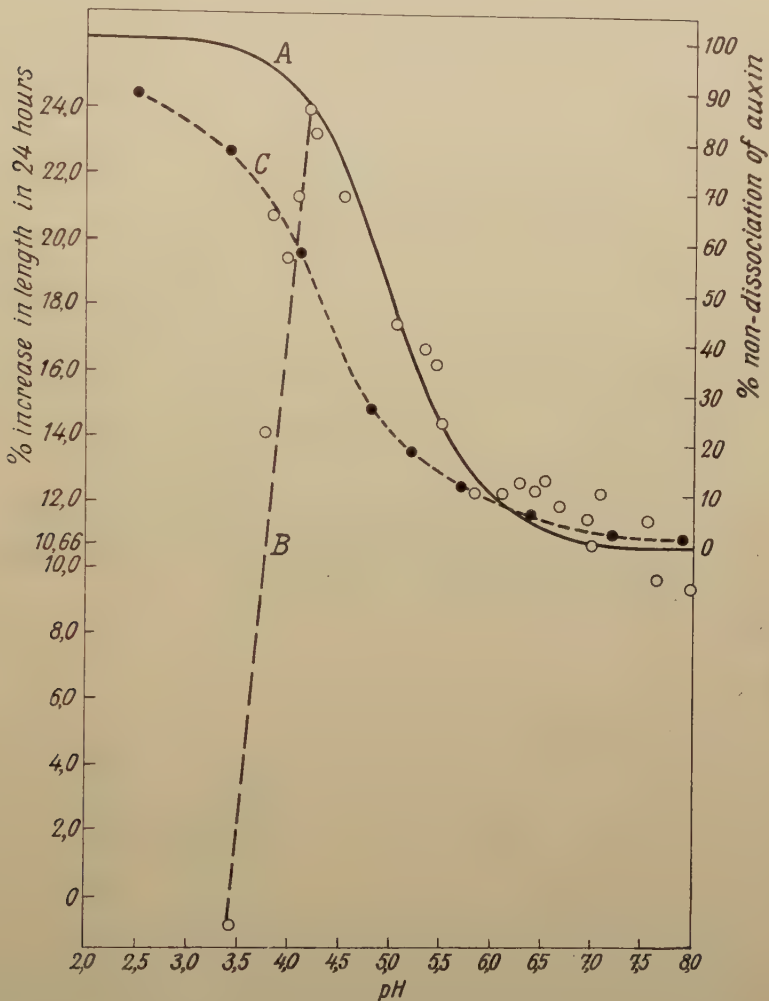


Fig. 3. Variation of growth rate with pH.

- A. — titration curve of auxin (pK 5.0).
 B. o—o—o change of growth rate with pH.
 C. ●—●—● change of acid curvatures with pH (BONNER 1934).

The results of table I are plotted in open dots. Curve A is the theoretical dissociation curve of auxin. C is BONNER's curve for the variation of acid curvature with pH (see fig. 1), which clearly does not fit curve A. It appears that A is also the curve for the change of growth with pH, when both curves have been reduced to the same ordinate at pH 7.0. A critical point, however, is reached at pH 4.18. This is the

optimal pH for the growth of the *Avena* coleoptile. On the acid side of this point the growth rate shows a rapid fall.

BONNER reports that "external acidities of greater than pH 4.1 proved to be toxic after more than two hours". As he measured the acid curvatures after two hours this fact does not appear from his curves.

Likewise ALBAUM c.s. (1937) found that *Nitella* cells lost their turgor when they had been exposed for 30 minutes to acidities higher than pH 4.10.

In my experiments the sections have lost their turgor after they have been for one hour in a solution of pH 3.42. (fig. 4).

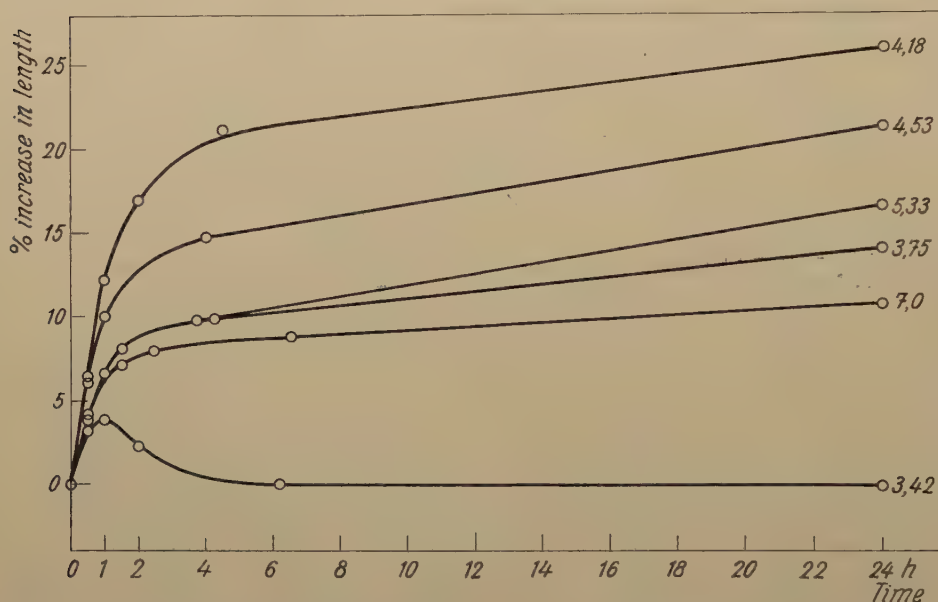


Fig. 4. Growth plotted against time for a few characteristic pH values.

One strongly gets the impression that the straight steep part of curve B represents the reversal of the sign of charge of the protoplasm colloids. The iso-electric point would then lie at about pH 3.75.

So it appears that the relation of hydrogen-ions to the growth rate of the *Avena* coleoptile may be rendered by a one-peaked curve.

From pH 7.0 to 4.18 the H-ions gradually reduce the dissociation of the available auxin and the increasing amount of free auxin acid causes increase of growth. Below pH 4.18 the cells are somehow damaged and growth stops entirely almost at once.

On the alkaline side of the neutral point hydroxyl-ions also inhibit growth, however to a much lower degree than do the hydrogen-ions.

Of peculiar interest is the zone from pH 5.8 to 6.5. Here we find a growth rate which is practically constant and always slightly higher than might be expected from the titration curve. Though I cannot quite explain

this phenomenon, it seems probable that the pH of the cell contents lies in this zone. In this case the small differences between the pH of the internal and external solution might not be capable of changing the pH of the former, better buffered, one.

If this holds true we would now know in a physiological way that the pH of the cell sap of *Avena* is about 6.15.

BONNER measured electrically the pH of crushed coleoptiles and found it to be 6.1. Crushing organs, however, always is an objectionable method for determining values in living cells.

If growth substance is added to the external solution, the relation between growth and pH remains essentially the same. A one-peaked curve results, which only lies as a whole on a higher level.

Table II shows the change of growth with pH in buffer solutions to which 3-indole acetic acid had been added in a concentration of 1 mgr/liter. Only the data about the critical points are given; those at other pH values correspond.

TABLE II. Variation of growth with pH in buffered solutions of hetero-auxin $1 \text{ in } 10^6$.

pH	% increase in length during 24 hours	
	hetero-auxin 10^{-6}	no hetero-auxin
4.18	37.7	23.9
7.0	25.5	10.66

It is evident that in the *Avena* coleoptile the actions of acid and growth substance are simply accumulative.

The same was found by THIMANN (1935) who first caused sections to consume practically all their available growth substance in an acid buffer and afterwards put them into solutions of 3-indole acetic acid, whereupon they resumed their growth. I repeated those experiments and can fully confirm them.

More recently MARMER (1937) has published data which indicate that one may obtain an increased inhibition of root growth either by lowering the pH at a constant concentration of hetero-auxin or by increasing the concentration of growth substance at constant pH.

Regarding the results of MEESTERS (1936) and LANE (1936) we may mention that both investigators used non-buffered solutions for their experiments, which cannot be expected to considerably change the pH of the extremely well buffered cell contents. Besides MEESTERS only considered a pH area from 4 to 5.2, which is not extensive enough as was shown in the above experiments.

It seems urgent to reconsider these results, especially since several investigators (FIEDLER 1936, AMLONG 1936, GEIGER-HUBER and BURLET

1936) found that growth substance in very low concentrations may stimulate growth of roots.

It is improbable that the relation of growth to pH in other organs is fundamentally different from that in the *Avena* coleoptile.

Experiments with roots and stems of other plants are being made with the same technique as described above and will be published before long.

ALBAUM, KAISER and NESTLER (1937) studied the rate of penetration of 3-indole acetic acid into *Nitella* cells as related to the pH of the external solution. They showed that the cell sap is such a well buffered system that its pH cannot be changed from the outside in the course of one hour. From this we may not conclude, however, as the authors seem to do, that it is impossible to change the acidity of the cell contents.

Cells gradually adapt themselves to the pH of the environment, if only they are long enough exposed to it and if it is not toxic from the start. This shows clearly from my experiments.

ALBAUM c.s. could demonstrate that the curve representing the velocity of penetration as a function of pH coincides with the dissociation curve for 3-indole acetic acid. At a lower pH hetero-auxin penetrates into the cells more rapidly. This means that the substance enters as undissociated molecules. The authors suggest that the influence of pH on growth might be explained by these facts.

This, however, can never be the case in my experiments, nor in those of BONNER and of THIMANN, because in these experiments the auxin was only available *inside* the cells, while the acid solution was on the outside.

Only when we provide the cells with 3-indole acetic acid in an acid buffer solution the action may be a dual one. Firstly the undissociated hetero-auxin molecules penetrate more rapidly into the cell, secondly the hydrogen-ions enter the cell and convert the available auxin into its active form. Both processes result in an increase of growth.

Summary.

1. The growth of excised coleoptile sections of *AVENA* was studied in 0.01 molar phosphate buffer solutions of various pH. The solutions were continuously stirred by an air current.

2. It was not possible to obtain sections that were entirely free of auxin, either by deseeding or by decapitation.

3. The curve representing the growth rate as a function of pH coincides in part with the dissociation curve of auxin, indicating that the growth substance is only active in its undissociated acid form.

Solutions of higher acidity than pH 4.18 are injurious to the cells. Therefore a one-peaked curve results for the change of growth with pH with an optimum at pH 4.18.

4. The actions of hydrogen ions and of externally added 3-indole acetic acid are accumulated.

The investigations were carried out in the Botanical Laboratory of the State University, Utrecht. Acknowledgements are made to the director Prof. Dr. V. J. KONINGSBERGER for his criticism and constant interest.

Utrecht, April 15th, 1938.

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Médecine. — *Exaltation et maintien de la virulence, variation du taux d'agglutination, accroissement de la vitesse de déplacement et de propagation du B. coli de passage.* Par ONG SIAN GWAN.
(Communicated by Prof. W. A. P. SCHÜFFNER.)

(Communicated at the meeting of April 30, 1938).

Le *B. coli* peut, à la suite d'un ou plusieurs passages sur des lapins acquérir les propriétés suivantes:

1. *Augmentation de la virulence.*

Après passage sur lapins le *B. coli* devient en général plus virulent vis-à-vis de ces animaux. Ce résultat a été constaté sur plusieurs souches et en particulier sur les souches *D*, *K* et *R*. Exemple: en avril 1931 la dose mortelle de la souche *R* qui tue un lapin en 24 h. si l'on injecte par voie veineuse, est de $\frac{1}{4}$ d'une culture de 24 h. sur gélose. Huit mois et demi plus tard la dose mortelle est devenue les $\frac{7}{10}$ d'une même culture. À la suite de 3 passages sur lapins la dose mortelle est devenue $\frac{1}{5}$ d'une culture sur gélose, mais la virulence de cette souche de passage *R*₃ s'affaiblit et en janvier 1933 la dose mortelle est devenue une demi-culture de 24 h. sur gélose.

Remarquons que depuis, la virulence s'est maintenue constante pendant au moins 4 ans (janvier 1933—1937). Le même résultat a été obtenu avec la souche *K*₅ (souche du 5e passage); en effet depuis décembre 1932 jusqu'au dernier titrage en janvier 1937 la dose mortelle pour un lapin en injection intraveineuse est égale aux $\frac{3}{10}$ d'une culture de 24 h. sur gélose. Toutes les souches sont conservées à la glacière (température 5°) dans le sang d'un lapin mis en tube scellé. Dans cette condition le *B. coli* conserve sa vitalité au moins sept ans; en effet trois souches de *B. coli* *D*₄, *D*₅ et *D*₆ mises en tubes au mois de mai et de juin en 1930 donnent encore une culture positive en juillet 1937.

L'exaltation de la virulence par passage sur animal est connue depuis 1869 par les expériences de COZE et FELTZ ¹⁾ et surtout par les travaux de PASTEUR, CHAMBERLAND et ROUX ²⁾. Depuis on a obtenu le même résultat avec d'autres microbes pathogènes et même saprophytes (CHARRIN et de NITTIS ³⁾; H. VINCENT ⁴⁾. On admet en général que le microbe perd facilement sa virulence dès qu'on cesse le passage sur animal. Les expériences rapportées ci-dessus représentent une exception à cette règle.

¹⁾ E. DUCLAUX. Le microbe et la maladie. Paris, p. 175 (1886).

²⁾ C. R. Ac. Sc., 92, 429 (1881).

³⁾ C. R. Soc. Biol., 49, 711 (1897).

⁴⁾ Ann. Inst. Pasteur, 12, 785 (1898).

2. Variation du taux d'agglutination.

Les tableaux 1 et 2 montrent les résultats d'agglutination des souches de *B. coli*: D_n , K_n et R_n vis-à-vis de deux sérums agglutinants différents: sérum anti K_5 et anti R_3 . Les souches K_5 et R_3 utilisées pour la préparation

TABLEAU 1.
Taux d'agglutination du sérum anti R_3 vis-à-vis du *B. coli* de passage.

Souches de <i>B. coli</i>	m	Souches de <i>B. coli</i>	m	Souches de <i>B. coli</i>	m
R_0	11	D_0	9	K_0	13
R_1	10	D_1	8	K_1	7
R_2	10	D_2	7	K_2	7
R_3	10	D_3	6	K_3	7
R_4	10	D_4	6 ¹⁾	K_4	6
		D_5	6 ¹⁾	K_5	5
		D_6	5	K_6	4

Taux d'agglutination = $1/10.2^m$.

TABLEAU 2.
Taux d'agglutination du sérum anti K_5 vis-à-vis du *B. coli* de passage.

Souches de <i>B. coli</i>	m	Souches de <i>B. coli</i>	m	Souches de <i>B. coli</i>	m
K_0	3	R_0	3	D_0	5
K_1	3	R_1	3	D_1	5
K_2	8	R_2	3	D_2	5
K_3	7	R_3	3	D_3	6
K_4	8	R_4	3	D_4	5
K_5	8			D_5	5
K_6	6			D_6	6

Taux d'agglutination = $1/10.2^m$.

du sérum sont les souches couramment employées dans ce travail. Les indices, 1, 2, ..., n désignent le nombre de passages sur lapin. On peut en conclure que la souche R_0 conserve son taux d'agglutination après 1 à 4 passages aussi bien pour le sérum anti R_3 , que pour le sérum anti K_5 . Le même résultat a été obtenu avec la souche D_0 si l'on utilise le sérum anti K_5 (tableau 2). Par contre le taux d'agglutination diminue progressivement après chaque passage si l'on emploie le sérum anti R_3 (tableau 1).

¹⁾ Agglutination faiblement positive.

Le taux d'agglutination du sérum anti R_3 vis-à-vis de la souche K_0 diminue après le passage (tableau 1); par contre il augmente brusquement entre K_1 et K_2 , si l'on utilise le sérum anti K_5 . La valeur d'agglutination demeure constante (1/2500) après d'autres passages. Le taux d'agglutination du sérum vis-à-vis de la souche employée (R_3 ou K_5) est supérieur ou égal au taux d'agglutination du même sérum vis-à-vis d'autres souches, c'est là un fait déjà connu. La souche K_0 constitue une exception, le taux d'agglutination du sérum anti R_3 vis-à-vis de cette souche (1/10 . 2^{13}) est supérieur au taux vis-à-vis de la souche R_3 employée (1/10 . 2^{10}).

D'autres facteurs inconnus jouant un rôle, ces résultats doivent être considérés avec une certaine réserve. En effet, en inoculant la dose mortelle de la souche R_3 (1/10 d'une culture de 24 h. sur gélose) dans le péritoine de sept cobayes, on trouve en employant le même sérum anti R_3 (titre 1/10000) des taux d'agglutination variant de 1/40 à 1/5000 vis-à-vis des *B. coli* isolés du pus péritonéal. La souche R_3 conserve son agglutinabilité non seulement après passages sur lapins, comme on vient de l'indiquer, mais encore après 31 passages sur gélose.

H. VINCENT ¹⁾ a remarqué que le *B. mégaterium* cultivé en sac de collodion dans le péritoine d'un cobaye est moins agglutiné (1/10) qu'avant le passage (1/20) par le même sérum agglutinant. Un résultat semblable a été obtenu par VAN LOGHEM ²⁾: certaines souches de *B. coli* peuvent perdre leur agglutinabilité après plusieurs passages sur gélose ou après avoir été cultivées en sac de collodion dans le péritoine d'un cobaye.

3. *Accroissement de la vitesse de déplacement.*

On désigne par vitesse de déplacement la projection horizontale de la distance moyenne parcourue en une seconde par le microbe. On la mesure à l'aide d'un microscope et de la cuve plate, employée généralement pour la numération globulaire (cyto-hématomètre). On introduit dans la cuve 1 à 2 gouttes d'une culture en bouillon de *B. coli* âgée de 2 à 4 h. On la recouvre d'une lamelle de verre. Une chambre claire est adaptée au microscope. Celui-ci a une position verticale, la préparation une position horizontale, on mesure donc la projection horizontale du déplacement du microbe. L'objectif employé a un fort grossissement sans immersion (objectif Leitz 7, oculaire 4). Le trajet parcouru est dessiné sur une feuille de papier, en même temps que le temps est mesuré au chronomètre. On connaît la longueur s du chemin parcouru en μ ($= 1/1000$ mm) ainsi que le temps t en secondes; on en déduit la vitesse de déplacement: $v = \frac{s}{t}$.

Pour ces mesures on choisit des microbes possédant une mobilité différente et on prend la moyenne d'une vingtaine de mesures au moins.

Le tableau 3 donne les résultats obtenus avec les souches R , D et K et

¹⁾ Ann. Inst. Pasteur, 12, 785 (1898).

²⁾ Nederl. Tijdschr. v. Geneesk., II, 1966 (1921).

leurs dérivés après le passage R_n , D_n et K_n , l'indice n désigne le nombre de passages sur lapins. On constate que la vitesse de déplacement augmente

TABLEAU 3.
Vitesse de déplacement v en μ /sec du B. coli de passage.

Souches de B. coli	v	Souches de B. coli	v	Souches de B. coli	v
R_0	22	D_0	23	K_0	20
R_1	31	D_1	30	K_1	33
R_2	30	D_2	30	K_2	31
R_3	32	D_3	24	K_3	22
R_4	35	D_4	23	K_4	31
		D_5	22	K_5	28
		D_6	23	K_6	31
				K_7	28
				K_8	37

en général après passage sur l'animal, mais cet accroissement n'est pas constant. Dans ces expériences les mesures ayant été faites 3 ans après les passages, il est probable que le résultat serait plus uniforme, si l'on pratiquait ces mesures peu de temps après le passage. La mesure de la vitesse de déplacement de la souche K_8 a été faite peu après le passage, elle a donné une vitesse maxima : $v = 37 \mu$ /sec. Il n'y a pas de relation entre la vitesse de déplacement et la virulence de deux souches différentes. En effet les souches D_6 et K_7 sont plus virulentes que la souche R_3 , mais leurs vitesses ne sont pas plus grandes. La souche S_0 et la souche D_u ont la même vitesse de déplacement: $v = 18 \mu$ /sec; la première tue le cobaye en 24 h. si l'on injecte dans le péritoine 1/5000 d'une culture de 24 h. sur gélose. La deuxième ne le tue qu'à la dose d'un dixième d'une culture sur gélose, dose qui est 500 fois plus grande que la précédente. On verra plus bas qu'on peut augmenter in vitro la vitesse de déplacement du B. coli.

Une remarque concernant le mouvement et la vitesse de déplacement du B. coli. La vitesse de déplacement ne dépend pas toujours de la grosseur et de la longueur du bacille. Il existe des bacilles, qui se déplacent en ligne droite avec une énorme vitesse, difficile à mesurer, d'autres suivent cette même ligne droite en effectuant des mouvements ondulatoires.

Il n'y a pas de direction préférée. En effet un bacille qui se dirige tout droit peut s'arrêter tout à coup sans cause apparente et prendre ensuite, sans s'être retourné la direction opposée.

A la suite d'un choc contre un autre bacille, il s'arrête et semble étourdi, ou bien marche en sens inverse sans s'arrêter et sans se retourner. A côté

des bacilles qui suivent une ligne droite, il en est qui décrivent des cercles ou des courbes quelconques. D'autres, en tournant très vite sur eux-mêmes comme les ailes d'un moulin, avancent en ligne droite ou bien décrivent un cercle ou une courbe quelconque.

Est-ce un phénomène vital ou physico-chimique? On sait que certaines substances, par exemple une parcelle de camphre sur l'eau, effectue le même mouvement et on explique le phénomène par une différence de tension superficielle.

4. *Accroissement de la vitesse de propagation.*

Un *B. coli* ou un autre microbe mobileensemencé en tube à la surface d'un milieu de culture demi-solide (contenant 2 p 1000 de gélose) donne en fonction du temps une culture, qui se propage de haut en bas vers le fond du tube. Ce phénomène a été observé indépendamment l'un de l'autre par WASSÉN (1923) ¹⁾ et DIMITRIJEVIC-SPETH ²⁾. Remarquons qu'il se produit aussi bien dans un milieu de culture liquide (bouillon Martin), si l'on prend un tube ayant un petit diamètre: 3 à 5 mm. Dans toutes ces expériences on a utilisé le milieu demi-solide préconisé par FISCHER ³⁾ (gélose 2 p. 1000, peptone Chapoteaut 2 p. 100, pH 7.5). Dans ce milieu de culture le *B. coli* donne deux types différents de culture. 1. Un voile qui se propage de haut en bas. 2. Un disque opaque qui se forme au bout de 2 à 3 hrs. descendant de haut en bas. Ce disque opaque est nettement limité à la partie inférieure. Au-dessus du disque le milieu de culture se clarifie. Quelquefois on observe au-dessus de ce disque un ou plusieurs autres disques et FISCHER a distingué dans ce groupe deux types de cultures, suivant que la culture a donné un ou 2 disques. On a même observé jusqu'à 5 disques superposés et séparés les uns des autres par une zone claire. La plupart des souches employées ne donnent que deux disques. Pour ces expériences on tient compte seulement du disque qui se trouve le plus bas.

A. *Expression de la courbe de propagation.*

Désignons par y la distance mesurée en mm de la surface du milieu de culture à la partie inférieure du disque de culture et par t le temps en heures. On obtient une courbe expérimentale $y = f(t)$ partant de l'origine 0, dont la concavité est tournée vers le haut (fig. 1). Cette courbe peut être convenablement représentée par l'équation:

$$y = at^b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

où a et b sont deux constantes positives et en général $b > 1$ ⁴⁾. Cette

¹⁾ C. R. Soc. Biol., 104, 523 (1930).

²⁾ Zentralbl. f. Bakt. I, O., 93, 486 (1924).

³⁾ Acta pathologica et microbiologica scandinavica. Suppl. IX (1932).

⁴⁾ Il est curieux de noter l'identité de l'équation (1) avec la formule d'absorption de FREUNDLICH (1907): $a = \alpha c^{1/n}$, qui trouve également son application biologique, dans

prendre une forme linéaire (2). En effet, pour $t > 5$ on a dit plus haut que (2)

$$y = a_1 t + b_1$$

représente mieux le phénomène. Prenons les logarithmes:

$$\log y = \log(a_1 t + b_1) = \log \left[a_1 \left(t + \frac{b_1}{a_1} \right) \right] = \log a_1 + \log \left(t + \frac{b_1}{a_1} \right)$$

et le graphique logarithmique ($\log y$, $\log t$) n'est plus linéaire, mais il ne s'écarte pas beaucoup de la ligne droite. Au point $t=0$, $y=0$ la tangente à la courbe est horizontale; on a en effet $y' = abt^{b-1} = 0$.

Supposons la fonction $y = at^b$ continue à partir de $t=0$, y' croît en même temps que t , par conséquent la concavité de la courbe est tournée vers le haut, comme il a été constaté pour la courbe expérimentale. (fig. 1).

Afin de vérifier la formule empirique (1) on a choisi parmi de nombreuses expériences la suivante: trois tubes (longueur 45 cm, diamètre 1.5 cm) contenant le milieu de culture demi-solide indiqué plus haut, sont ensemencés à la surface avec le *B. coli* So. On colle derrière le tube une bande de papier millimétré, indiquant la distance à partir de la surface du milieu de culture. On place les tubes à l'étuve, on fait toutes les 15 min. la lecture à travers la vitre de l'étuve à l'aide d'une loupe et on note en même temps la température dans l'étuve.

De cette manière on évite les brusques changements de température qui pourraient influencer la vitesse de propagation lorsque l'on fait des lectures rapprochées. La température moyenne pendant l'expérience est de $33^{\circ},9 \pm 0.53$. Dans la tableau 4 on trouve les données du tube 1 et le graphique logarithmique (fig. 2) est construit avec ces données. Comme on l'a vu, la courbe expérimentale $y = f(t)$ se compose de deux parties.

La première partie correspond à l'équation (1) $y = at^b$, t varie de 0 à 4.5. Le calcul des constantes a et b est fait en remplaçant dans l'équation (3) Y et T par les valeurs observées correspondantes Y_K et T_K ; puis on range les équations approchées dans l'ordre des valeurs croissantes de T . Ensuite on divise les équations en deux groupes contenant un nombre d'équations égal ou à une unité près. On fait la somme des équations de chaque groupe, on obtient ainsi deux équations du premier degré en b et A dont les solutions sont les valeurs cherchées de A , b et a (méthode des groupements)¹⁾. On trouve ainsi pour cette courbe: $A = \log a = 0.418$ ou $a = 2.62$ et $b = 1.89$.

L'écart arithmétique moyen entre les valeurs logarithmiques observées $\log y_{\text{obs.}}$ et les valeurs logarithmiques calculées $\log y_{\text{calc.}}$ est égal à:

$$\omega = \frac{\sum |\log y_{\text{obs.}} - \log y_{\text{calc.}}|}{n} = \frac{0.211}{15} = 0.014$$

¹⁾ FRÉCHET et ROMANN. Représentation des lois empiriques par des formules approchées. 1930. Eyrolles, édit, p. 71, Paris.

TABLEAU 4.

 t = temps en heures. $y_{\text{obs.}}$ = distance mesurée en mm de la surface du milieu de culture à la partie inférieure du disque de culture.

t	$y_{\text{obs.}}$	$\log t$	$\log y_{\text{obs.}}$	$\log y_{\text{calc.}}$	$ \log y_{\text{obs.}} - \log y_{\text{calc.}} $	$y_{\text{calc.}}$	$ y_{\text{obs.}} - y_{\text{calc.}} $	Temp.
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première partie : $y = at^b$

1.00	3	0.0000	0.4771	0.418	0.059	2.62	0.38	34°5
1.25	4	0.0969	0.6020	0.601	0.001	3.99	0.01	34°5
1.50	5	0.1760	0.6989	0.751	0.052	5.64	0.64	34°5
1.75	8	0.2430	0.9030	0.877	0.026	7.54	0.46	34°5
2.00	9	0.3010	0.9542	0.987	0.033	9.71	0.71	34°2
2.25	12	0.3521	1.0791	1.082	0.003	12.07	0.07	34°2
2.50	15	0.3979	1.1760	1.170	0.006	14.79	0.21	34°2
2.75	17.5	0.4393	1.2430	1.249	0.006	17.74	0.24	34°2
3.00	21	0.4771	1.3222	1.320	0.002	20.90	0.10	34°2
3.25	24	0.5118	1.3802	1.386	0.006	24.30	0.30	34°2
3.50	28	0.5440	1.4471	1.446	0.001	27.93	0.07	34°2
3.75	32	0.5740	1.5051	1.503	0.002	31.85	0.15	34°2
4.00	36	0.6020	1.5563	1.556	0.000	35.98	0.02	34°2
4.25	40	0.6283	1.6020	1.606	0.004	40.36	0.36	34°2
4.50	44	0.6532	1.6434	1.653	0.010	44.98	0.98	34°2

deuxième partie $y = a_1 t + b_1$

4.83	49	0.6842	1.6902			48.14	0.86	34°2
5.00	51	0.6989	1.7075			50.96	0.04	34°2
5.25	55	0.7201	1.7403			55.11	0.11	34°2
5.50	59	0.7403	1.7708			59.26	0.26	34°4
6.00	67	0.7781	1.8260			67.55	0.55	34°8
6.50	76	0.8129	1.8808			75.85	0.15	31°6
7.00	84	0.8451	1.9242			84.14	0.14	32°5
7.58	94	0.8798	1.9731			93.76	0.26	33°5
8.00	100.5	0.9030	2.0022			100.73	0.23	33°9

et celui entre les valeurs observées $y_{\text{obs.}}$ et les valeurs correspondantes calculées $y_{\text{calc.}}$ est égal à:

$$\varepsilon = \frac{\sum |y_{\text{obs.}} - y_{\text{calc.}}|}{n} = \frac{4,70}{15} = 0,31.$$

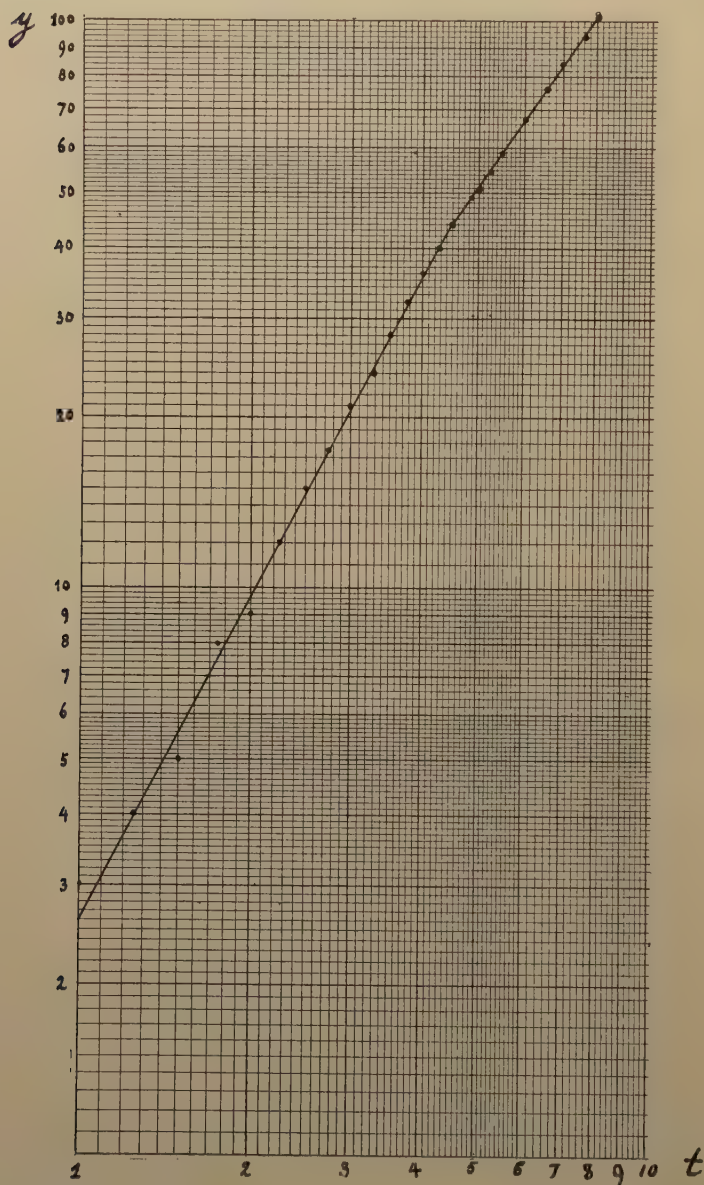


Fig. 2. Courbe de propagation du *B. coli* So. en coordonnées logarithmiques.

L'équation (1) représente donc assez bien la courbe expérimentale étant donné que l'erreur absolue commise sur la mesure de y est de ± 0.5 mm.

L'écart ω mesuré sur le graphique logarithmique est sensiblement propor-

tionnel à l'écart relatif $\frac{\varepsilon}{|y_{\text{obs.}}|}$. En effet, posons $\omega = d \log y_{\text{obs.}}$, on a:

$$d \log y_{\text{obs.}} = \log e \frac{dy_{\text{obs.}}}{|y_{\text{obs.}}|} = 0,434 \frac{\varepsilon}{|y_{\text{obs.}}|}.$$

L'interpolation par le graphique logarithmique permet donc de calculer les valeurs de y avec une erreur relative $\frac{0,434 \varepsilon}{|y_{\text{obs.}}|}$.

La deuxième partie de la courbe expérimentale (fig. 1) correspond à une droite (2) $y = a_1 t + b_1$; le calcul des constantes par la méthode des groupements donne: $a_1 = 16.59$ et $b_1 = -31.99$ avec un écart arithmétique moyen entre les valeurs observées $y_{\text{obs.}}$ et les valeurs correspondantes calculées $y_{\text{calc.}}$.

$$\varepsilon_1 = \frac{\sum |y_{\text{obs.}} - y_{\text{calc.}}|}{n} = \frac{2,59}{9} = 0,29.$$

L'équation (2) représente donc assez bien la courbe expérimentale pour $t > 4.5$. Dans d'autres expériences les valeurs y_k et t_k ont été mesurées jusqu'à $t = 27.5$ et les équations (1) et (2) représentent de manière aussi satisfaisante la courbe obtenue. Le calcul des constantes A et b par la méthode graphique donne: $A = 0.41$ et $b = 1.98$, qui correspondent aux valeurs calculées par la méthode des groupements: $A = 0.418$ et $b = 1.89$.

FISCHER¹⁾ a obtenu une courbe expérimentale semblable, mais il situe la région linéaire à partir de $t = 6$ ou 7 , tandis qu'on a constaté et calculé au moyen de l'équation (1) que cette région semble commencer pour $t \cong 5$.

A quoi correspond la région linéaire (2) de la courbe expérimentale? Il est probable, comme M. J. RÉGNIER nous l'a suggéré, que le manque d'oxygène au fond du tube joue un rôle. Pour le démontrer, on verse le milieu nutritif demi-solide dans une boîte de PÉTRI, où on a collé un papier millimétré. On ensemente le centre de la boîte de PÉTRI avec un *B. coli* mobile et on mesure à partir du centre les distances franchies par la culture dans les quatre directions perpendiculaires. La courbe expérimentale ainsi obtenue ne devient plus linéaire pour $t > 5$ et l'équation (1) semble donner dans cette région une meilleure approximation que l'équation (2). En effet, les expériences exécutées dans ces conditions avec deux souches de *B. coli* S_0 et Badin ont donné le résultat suivant: les écarts arithmétiques moyens ε entre les valeurs observées $y_{\text{obs.}}$ et les valeurs calculées $y_{\text{calc.}}$ par l'équation (1) sont inférieurs à ceux entre les valeurs observées $y_{\text{obs.}}$ et les valeurs calculées $y_{\text{calc.}}$ par l'équation (2): ε_1 (tableau 5). Il est à remarquer que les valeurs mesurées de y dans le tube vertical sont plus grandes que celles dans la boîte de PÉTRI. La pesanteur intervient-elle?

1) Acta pathologica et microbiologica scandinavica, suppl. IX (1932).

TABLEAU 5.

Souches de B. Coli	a	b	ε	a_1	b_1	ε_1
Badin.	0.95	1.88	1.15	7.75	-17.44	1.78
"	1.80	1.54	1.06	8.14	-21.14	1.13
"	1.32	1.71	1.07	8.61	-22.92	1.19
"	0.95	1.85	0.86	8.82	-26.68	0.91
"	2.15	1.59	0.64	11.25	-31.16	1.29
"	1.11	1.99	0.56	12.75	-37.03	0.69
So.	0.36	2.24	0.38	8.18	-29.28	0.40
"	0.36	2.24	0.39	8.35	-30.49	0.63
"	0.31	2.31	0.33	8.88	-34.18	0.44
"	0.28	2.36	0.38	8.52	-31.71	0.50

Le manque d'oxygène ne semble pas devoir être le seul facteur intervenant ici. Il est probable que l'intoxication du milieu de culture par les produits métaboliques et la vitesse de multiplication du microbe jouent un rôle.

JENSEN ¹⁾ a montré qu'une culture de B. coli donne 4 phases distinctes de croissance. Dans la première phase (période latente), qui dure 1 à 2½ heures, le microbe ne se multiplie pas ou commence à se multiplier à la fin de la période.

La deuxième phase est la période de multiplication maxima. Soient a et b les nombres respectifs de microbes au début et après n générations, on a une fonction exponentielle: $b = a \cdot 1,6^n$. Cette période dure 2 à 4 h. pour le B. coli.

Dans la 3e phase la vitesse de multiplication diminue et enfin dans la 4e phase la mortalité des microbes est plus grande que la natalité. La première et la deuxième phase peuvent donc durer 3 à 6½ h. et en moyenne 4 h. 45 min.

M. RÉGNIER et Mlle LAMBIN ²⁾ ont également montré que ces deux périodes durent six heures.

Ou a vu qu'entre 0 et 5 heures la courbe de propagation peut être représentée par l'équation (1).

La période de multiplication maxima correspond à la vitesse maxima de propagation. Après cette période la vitesse de multiplication diminue parallèlement à la vitesse de propagation; en effet la courbe semble prendre une forme linéaire. Remarquons que la vitesse de déplacement ne varie pas

¹⁾ Undersøgelser over colibacillens normale Vaekstforhold. Disputats København. 1927 (in FISCHER).

²⁾ C. R. Ac. Sc., 199 (1934) séance du 10 décembre.

beaucoup pendant la propagation. Voici les moyennes des mesures de vitesse de déplacement en $\mu/\text{sec.}$, faites toutes les heures en milieu demi-solide à partir de deux heures après le repiquage: 18.4, 18.2, 16.3, 13.1, 16.2, 15.4, 16.8.

B. *Courbe de propagation du B. coli de passage.*

Dans la figure 3 on a porté sur l'axe des ordonnées la distance y en mm de la surface du milieu de culture à la partie inférieure du disque de culture et sur l'axe des abscisses le temps t en heures. On constate qu'après chaque passage la pente de la courbe expérimentale $y = f(t)$ devient de plus en plus grande: la vitesse de propagation est augmentée. Le tableau 6 donne les valeurs des constantes a et b de l'équation $y = at^b$ appliquée

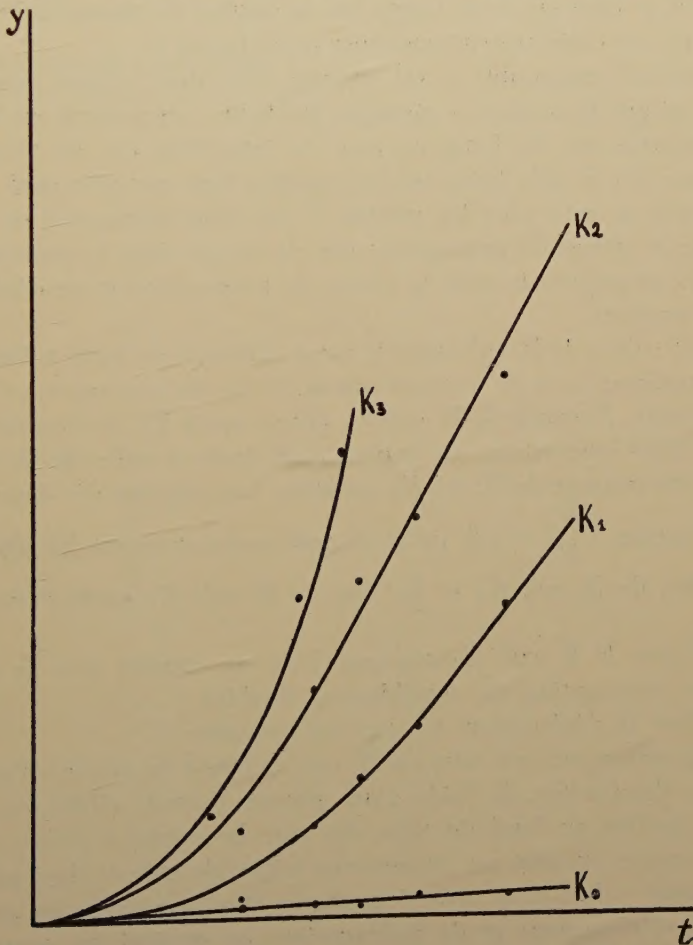


Fig. 3. Courbes de propagation du *B. coli* après passage sur lapin.
 K_0 = souche d'origine.

K_1 , K_2 et K_3 souches après le 1e, 2e et 3e passage.

aux données du *B. coli* de passage K_n , t varie de 0 à 5.5. Les valeurs de a et b de la souche K_0 ne sont que relativement exactes, car les mesures

de y ne sont pas précises. On constate que a augmente avec le nombre de passages, par contre b diminue. La 4e colonne donne la pente y' calculée

TABLEAU 6.

Souches de B. coli	a	b	y'
K_0	(0.63)	(0.76)	$19^{\circ},0$
K_1	0.03	3.89	$81^{\circ},1$
K_2	0.46	2.83	$86^{\circ},5$
K_3	1.07	2.59	$87^{\circ},7$

pour $t = 4$, y' croit en même temps que le nombre de passages; ce résultat correspond aux courbes expérimentales de la figure 3.

Un résultat comparable a été obtenu avec deux autres souches D_n et R_n , n désigne le nombre de passages sur lapin. Les mesures de la vitesse de propagation ont été faites au mois de juin 1932, peu de temps après les passages du B. coli. Elles ont été répétées trois ans plus tard en 1935; les résultats ne sont plus les mêmes. Il est donc nécessaire de faire la mesure de la vitesse de propagation peu de temps après le passage.

On peut augmenter in vitro la vitesse de propagation et de déplacement de deux manières.

10. On cultive le B. coli dans le sang défibriné de lapin à l'étuve. Au bout de quelques jours on constate que la vitesse de propagation du B. coli est augmentée. Exemple: le B. coli K_4 donne après 9 j. de contact avec le sang défibriné une vitesse de propagation égale à celle du B. coli K_5 , obtenu après passage du B. coli K_4 sur lapin. Les rapports des deux vitesses de propagation: $\frac{V_5}{V_4} = 1.5$, v_4 et v_5 sont respectivement les vitesses de propagation de B. coli K_4 et K_5 (ou de B. coli K_4 après contact avec le sang).

Si on laisse le B. coli K_4 pendant 11 j. en contact avec le sang, la vitesse de propagation est sensiblement doublée.

La vitesse de déplacement n'a pas été mesurée.

20. On ensemence en tube un B. coli mobile à la surface d'un milieu de culture demi-solide. A l'aide d'une pipette Pasteur effilée, on prélève quelques gouttes au fond du tube, dès que la culture a atteint celui-ci. On l'ensemence aussitôt sur un nouveau milieu de culture demi-solide, on répète plusieurs fois cette opération. Après chaque passage on mesure la vitesse de déplacement et de propagation; en général on constate une augmentation. Voici les moyennes des mesures de vitesses de déplacement: v_n après des passages in vitro en μ/sec . Les indices 0, 1, 2 ..., n désignent le nombre de passages dans le milieu demi-solide: $v_0 = 17.5$; $v_1 = 27.7$; $v_2 = 24.2$; $v_3 = 23.5$; $v_6 = 28.7$; $v_9 = 28.2$; $v_{10} = 29.6$; $v_{11} = 31.8$; $v_{12} = 33.9$.

On constate qu'en partant d'une souche de vitesse moyenne $v_0 = 17.5$ μ /sec. on obtient après le 12^e passage in vitro une vitesse moyenne de déplacement $v_{12} = 33.9$ μ /sec. valeur presque deux fois plus grande. D'autres souches de *B. coli* ont donné un résultat semblable.

FISCHER a observé que le passage dans le milieu demi-solide fait augmenter la vitesse de propagation. BUONOMINI ¹⁾ a trouvé un accroissement de mobilité de *B. coli*, qui a traversé des bougies poreuses.

Ce sont les faits tels que nous les avons mis en évidence. Plusieurs interprétations peuvent être données de ce phénomène. Bornons-nous à citer l'une d'elles, plus particulièrement à l'ordre du jour: l'hypothèse que ces variations de virulence, de vitesse de propagation et de déplacement peuvent être en relation avec ce qu'on est convenu d'appeler les „mutations microbiennes”.

¹⁾ Lo sperimentale, **5**, 608 (1934).
Bull. Inst. Pasteur, **34**, 1125 (1936).

